

# On morphisms killing weights, weight complexes, and Eilenberg-MacLane (co)homology of spectra

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## Abstract

The main goal of this paper is to study when a morphism  $g$  in a triangulated category  $\underline{\mathcal{C}}$  endowed with a weight structure "kills certain weights" of objects (between an integer  $m$  and some  $n \geq m$ ). If  $g = id_M$  (where  $M \in \text{Obj } \underline{\mathcal{C}}$ ) and  $\underline{\mathcal{C}}$  is Karoubian, then  $g$  kills weights  $m, \dots, n$  if and only if there exists a (weight) decomposition of  $M$  that *avoids* these weights (in the sense earlier defined by J. Wildeshaus).

We prove the equivalence of several definitions for killing weights. In particular, we describe a family of cohomological functors that "detects" this notion. We also prove that  $M$  is *without weights*  $m, \dots, n$  (i.e., a decomposition of  $M$  avoiding these weights exists) if and only if this condition is fulfilled for its *weight complex*  $t(M)$ .

These results allow us to get new (stronger) results on the conservativity of the weight complex functor  $t$ . We study in detail the case  $\underline{\mathcal{C}} = SH$  (endowed with the *spherical weight structure* whose heart consists of coproducts of sphere spectra); the corresponding weight complex functor is just the one calculating the  $H\mathbb{Z}$ -homology (whereas the terms of weight complexes are free abelian groups). In this case  $g$  kills weights  $m, \dots, n$  if and only if  $H(g) = 0$  for all  $H$  represented by elements of  $SH[m, n]$  (and  $g$ 's satisfying these conditions form an injective class in the sense defined by J.D. Christensen; yet this class

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is not stable with respect to shifts). Besides, for any spectrum  $M$  there exists a "weakly universal decomposition"  $P \rightarrow M \rightarrow I_0$  for  $I_0 \in SH[m, n]$  and  $P$  being without weights  $m, \dots, n$  (the latter condition has an easy description in terms of homology); thus we obtain a *complete Hom-orthogonal pair* in the sense defined by Pospisil and Stovicek. Our results can also be applied to (the usual) derived categories.

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## Introduction

Recall that a weight structure  $w$  (as independently defined in [Bon10a] and in [Pau08]) is a tool for endowing objects of a triangulated category  $\underline{C}$  with certain weight filtrations; these filtrations yield functorial ("weight") filtrations and spectral sequences for any (co)homology of these objects. In particular, if  $w$  is (a version of) the Chow weight structure (as constructed in [Bon10a], [Heb11], [Bon14], or [BoI15]) for a certain motivic category then we obtain Deligne's weights for étale and singular cohomology; the weight structure theory also yields many more interesting filtrations and spectral sequences (including the Atiyah-Hirzebruch ones for the cohomology of spectra).

In a series of papers J. Wildeshaus has studied motives *without weights*  $m, \dots, n$  for  $m \leq n \in \mathbb{Z}$ . Those are motives fitting into distinguished triangles of the form  $X \rightarrow M \rightarrow Y$  for  $X$  of weight at most  $m - 1$  and  $Y$  of weight at least  $n + 1$  (i.e.,  $X \in \underline{C}_{w \leq m-1}$  and  $Y \in \underline{C}_{w \geq n+1}$  for  $w$  being the corresponding Chow weight structure on this motivic category  $\underline{C}$ ). Note that  $M$  determines its "components"  $X$  and  $Y$  functorially (in contrast to the case of "ordinary" weight decompositions where  $n = m - 1$ ); this yields a way of constructing "new" (and interesting) motives out of old ones. If  $M$  is without weights  $m, \dots, n$  (we write this as  $M \in \underline{C}_{w \notin [m, n]}$ ) then the corresponding factors of the weight filtration vanish for any cohomology of  $M$ . Somewhat amazingly, it is reasonable to expect the following interesting converse to this statement: Deligne's weights for étale cohomology "should detect" whether  $M \in \underline{C}_{w \notin [m, n]}$  (for motives with rational coefficients; some interesting cases of this conjecture were established in [Wil15b], [Wil15a], and [Wil09]).

In the current paper we (mostly) study the condition of being without weights  $m, \dots, n$  in arbitrary  $(\underline{C}, w)$ . Our main tool is the study of those morphisms that *kill these weights*, i.e., of  $g \in \underline{C}(M, N)$  "compatible with" certain morphisms  $w_{\leq n}M \rightarrow w_{\leq m-1}N$  (we write this as  $g \in \text{Mor}_{[\cancel{m}, \cancel{n}]} \underline{C}$ ). We prove that this definition of killing weights for  $g$  is equivalent to several other ones; in particular, killing weights is categorically self-dual (in a natural way).  $M$  is without weights  $m, \dots, n$  if and only if  $\text{id}_M \in \text{Mor}_{[\cancel{m}, \cancel{n}]} \underline{C}$  (if  $\underline{C}$  is Karoubian which is "usually" the case; we take this property for the definition of  $\underline{C}_{w \notin [m, n]}$  in general). Certainly, if  $g$  kills weights  $m, \dots, n$  then  $H(g)$  kills the corresponding factors for any (co)homology  $H$  on  $\underline{C}$ . The converse is also true if one considers all representable  $H$  here; unfortunately, étale cohomology is not sufficient for studying this property for motives.

Yet we describe a certain class of cohomology theories (on  $\underline{C}$ ) such that  $g \in \text{Mor}_{[m,n]} \underline{C}$  if and only if  $H(g) = 0$  for any theory belonging to this collection. For  $w$  being the *spherical* weight structure on  $SH$  (the stable homotopy category of spectra) one should take the theories represented by elements of  $SH[m, n]$  (in the notation of §3.2 of [Mar83], i.e., the homotopy groups of representing objects should vanish in all degrees  $\notin [m, n]$ ; see §2.4 below). If  $\underline{C} = DM$  (a "big" motivic category) then the general theory gives representing objects belonging to  $DM[m, n] = DM^{t_{Chow} \leq -m} \cap DM^{t_{Chow} \geq -n}$  (see Remark 2.2.4(3)); this is somewhat less satisfactory.

We also improve significantly our understanding of the weight complex functor  $t$  (from  $\underline{C}$  into a certain "weak category of complexes"  $K_w(Hw)$ ; weight complexes of this sort were defined in §3 of [Bon10a]) in this paper. Here  $Hw \subset \underline{C}$  is the additive category of objects of weight zero (i.e.,  $\text{Obj } Hw = \underline{C}_{w \leq 0} \cap \underline{C}_{w \geq 0}$ ); so it would be reasonable to expect that  $t$  (along with weight spectral sequences) "sees" the "finite weight part" of  $\underline{C}$  and ignores "infinitely small and infinitely large weights". Our current methods yield very precise statements of this sort; we also describe in detail the defect for  $t$  to "detect"  $\underline{C}_{w \leq 0}$  and  $\underline{C}_{w \geq 0}$  (see Theorem 2.3.4). Besides,  $t(M)$  sees whether  $M \in \underline{C}_{w \notin [m,n]}$ .

Now we discuss the relation of our definitions to the notion of an *injective class of morphisms* (that is dual to the notion of a *projective class* that was central in [Chr98]; injective classes are "somewhat closer" to  $\text{Mor}_{[m,n]} \underline{C}$  in our main examples). By definition (see Remark 2.2.4(2) below), any injective class can be described as  $\{f : H(f) = 0\}$  for  $H$  running through a certain family of cohomological functors  $\underline{C} \rightarrow Ab$ ; the same is true for  $\text{Mor}_{[m,n]} \underline{C}$  by Theorem 2.2.3(I). Yet in contrast to the definition of injective classes, we do not have to demand that all these  $H$  are representable (one may say that the existence of weight decompositions gives a substitute for this representability condition along with the corresponding version of the existence of enough injective objects; cf. §2.3 of [Chr98]). Note here that for  $\underline{C} = DM$  or  $\underline{C} = SH$  (and also for  $\underline{C} = D(\underline{A})$  for  $\underline{A}$  being an abelian category with enough projectives; see Remark 1.4.4(5)) the classes  $\text{Mor}_{[m,n]} \underline{C}$  are injective (since a  $t$ -structure adjacent to  $w$  exists in this case; see Remark 2.2.4(2)). Yet they are certainly not shift-stable in contrast to the main projective classes studied by J.D. Christensen (see the beginning of §3 of *ibid.*); this distinction of our focus of study from the one of *ibid.* is possibly even more important. One may say that our definition of  $\text{Mor}_{[m,n]} \underline{C}$  is more flexible and "takes into account filtrations". In particular, if a  $t$ -structure adjacent to  $w$  exists then for any

$l \geq 0$  the morphism class  $\cap_{m \in \mathbb{Z}} \text{Mor}_{[m, m+l]} \underline{C}$  is shift-stable and injective; yet one certainly cannot recover single  $\text{Mor}_{[m, m+l]} \underline{C}$ 's from this intersection. Respectively, no reasonable analogues of  $\underline{C}_{w \notin [m, n]}$  can be described using shift-stable injective classes of  $\underline{C}$ -morphisms. Still, our Theorem 2.2.1(3) is rather similar to Proposition 3.3 of *ibid.*<sup>1</sup> Besides, the aforementioned intersection construction yields an interesting (shift-stable) injective class in the case  $l = 0$  since  $\cap_{m \in \mathbb{Z}} \text{Mor}_{[m, m]} \underline{C}$  equals  $\{g \in \text{Mor } \underline{C} : t(g) = 0\}$ .

Furthermore, if a  $t$ -structure adjacent to  $w$  exists then for any  $M \in \text{Obj } \underline{C}$  there exists a "weakly universal decomposition"  $P \rightarrow M \rightarrow I_0$  for  $I_0 \in \underline{C}[m, n]$  (see Definition 1.4.3(II,III)) and  $P \in \underline{C}_{w \notin [m, n]}$ . Thus  $\underline{C}_{w \notin [m, n]}$  and  $\underline{C}[m, n]$  yield a *complete Hom-orthogonal pair* in the sense of Definition 3.2 of [PoS16] (see Remark 2.2.4(3) below for more detail). This is also an important notion; in particular, for  $\underline{C} = SH$  it may be interesting study the "interaction" of the ideals of morphisms characterized by the condition  $\pi_i(g) = 0$  (for  $i$  running through  $\mathbb{Z}$ ; the intersection of all these classes is the class of *ghost morphisms* defined in §7 of [Chr98]).

Now we describe the contents of the paper in more detail; some more information of this sort can also be found at the beginnings of sections.

In §1 we recall some basics on weight structures; only a few (somewhat technical) statements are new here. The reader not much interested in compactly generated triangulated categories (such as  $SH$ ), in projective and injective classes of morphisms, and in Hom-orthogonal pairs may probably ignore §1.4.

In §2 we introduce our main definitions of morphisms killing weights  $m, \dots, n$  and of objects without these weights. For  $\underline{C} = K^b(L - \text{vect})$  (or  $= K(L - \text{vect})$ ) and the stupid weight structure for this category we have  $g \in \text{Mor}_{[m, n]} \underline{C}$  if and only if  $H_i(g) = 0$  for  $-n \leq i \leq -m$  (in our numbering of homology and weights); yet the general definition is somewhat more complicated. We also prove several interesting properties of our notions; we sometimes demand  $\underline{C}$  to be Karoubian (i.e., if all idempotent  $\underline{C}$ -endomorphisms yield splittings of objects in it) in the formulations of this section. In particular, our notion of being without weights  $m, \dots, n$  in a non-Karoubian  $\underline{C}$  is somewhat more general than the one used by Wildeshaus. We closely relate our main notions to the weight complex functor  $t$ ; these results imply (in the Karoubian case) that  $t(g)$  is an isomorphism if and only if  $t(g)$  is an extension

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<sup>1</sup>Note that the proof of *loc. cit.* actually does not require shift-stability of the corresponding ideal.

of an object of "infinitely large weight" by a one of infinitely small weight. We (essentially) call extensions of the latter type *weight-degenerate objects* and study them in detail. We conclude the section by considering the case  $\underline{C} = SH$ ; for the corresponding *spherical* weight structure the weight complex functor yields complexes of free abelian groups computing the  $H\mathbb{Z}$ -homology of spectra, cellular filtrations of spectra yield their weight Postnikov towers, and weight spectral sequences are Atiyah-Hirzebruch ones.

We start §3 we use the results of [BoS16] to generalizing the results of the previous section to not (necessarily) Karoubian triangulated categories. One of these results is crucial for [Bon16]. We also construct certain examples illustrating the distinctions between the non-Karoubian and the Karoubian case.

Besides, we prove that we can "detect weights" of objects of  $\underline{C}$  using certain *pure* (co)homology. The arguments of the subsection dedicated to this subject are quite easy, and we include it in the paper for the purpose of applying the results in [BoT15]. Possibly, the author will prove some more complicated results of this sort in yet another paper (on the detection of weights and the study of Picard groups of various tensor triangulated categories).

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## 1 Notation and a reminder on weight structures

In this section we recall (a part of) the theory of weight structures.

In §1.1 we introduce some notation and conventions.

In §1.2 we recall some basics on weight structures. The only new statement of this section is Proposition 1.2.3(6) (it is rather technical but quite important for this paper).

In §1.3 we recall some properties of the weight complex functors. All of them except parts 3 and 5 of Remark 1.3.3 were established in [Bon10a].

In §1.4 we discuss the relation of weight structures to cohomology; so we recall the (somewhat more complicated) notions of weight filtrations, virtual  $t$ -truncations, and adjacent structures. The reader not much interested in compactly generated triangulated categories (such as  $SH$ ) and in injective classes of morphisms may ignore this section (along with the remarks that mention it later in the text) at the first reading of the paper.

## 1.1 Some notation and conventions

For categories  $C, D$  we write  $D \subset C$  if  $D$  is a full subcategory of  $C$ .

For a category  $C$ ,  $X, Y \in \text{Obj } C$ , we denote by  $C(X, Y)$  the set of  $C$ -morphisms from  $X$  to  $Y$ . We will say that  $X$  is a *retract* of  $Y$  if  $\text{id}_X$  can be factored through  $Y$ . Note that if  $C$  is triangulated or abelian then  $X$  is a retract of  $Y$  if and only if  $X$  is its direct summand.

For a category  $C$  we denote by  $C^{op}$  its opposite category.

For a subcategory  $D \subset C$  we will say that  $D$  is *Karoubi-closed* in  $C$  if it contains all retracts of its objects in  $C$ . We will call the smallest Karoubi-closed subcategory  $\text{Kar}_C(D)$  of  $C$  containing  $D$  the *Karoubi-closure* of  $D$  in  $C$ .

The *idempotent completion*  $\text{Kar}(\underline{B})$  (no lower index) of an additive category  $\underline{B}$  is the category of "formal images" of idempotents in  $\underline{B}$  (so  $\underline{B}$  is embedded into an idempotent complete category).

$\underline{C}$  and  $\underline{C}'$  will usually denote some triangulated categories. We will use the term *exact functor* for a functor of triangulated categories (i.e., for a functor that preserves the structures of triangulated categories).

For a distinguished triangle  $A \rightarrow B \rightarrow C$  in  $\underline{C}$  we will say that  $B$  is an *extension* of  $C$  by  $A$ .

We will say that a class  $D \subset \text{Obj } \underline{C}$  *strongly generates* a subcategory  $\underline{D} \subset \underline{C}$  and write  $\underline{D} = \langle D \rangle_{\underline{C}}$  if  $\underline{D}$  is the smallest strictly full triangulated subcategory of  $\underline{C}$  such that  $D \subset \text{Obj } \underline{D}$ . Certainly, here we can consider the case  $\underline{D} = \underline{C}$ .

For  $X, Y \in \text{Obj } \underline{C}$  we will write  $X \perp Y$  if  $\underline{C}(X, Y) = \{0\}$ . For  $D, E \subset \text{Obj } \underline{C}$  we write  $D \perp E$  if  $X \perp Y$  for all  $X \in D, Y \in E$ . For  $D \subset \text{Obj } \underline{C}$  we denote by  $D^\perp$  the class

$$\{Y \in \text{Obj } \underline{C} : X \perp Y \ \forall X \in D\}.$$

Dually,  ${}^\perp D$  is the class  $\{Y \in \text{Obj } \underline{C} : Y \perp X \ \forall X \in D\}$ .

We will say that certain  $C_i \in \text{Obj } \underline{\mathcal{C}}$  *weakly generate*  $\underline{\mathcal{C}}$  if  $\{C_i[j] : j \in \mathbb{Z}\}^\perp = \{0\}$ .

Below  $\underline{A}$  will always denote some abelian category;  $\underline{B}$  is an additive category.

In this paper all complexes will be cohomological, i.e., the degree of all differentials is  $+1$ ; respectively, we will use cohomological notation for their terms. We denote by  $K(\underline{B})$  the homotopy category of (cohomological) complexes over  $\underline{B}$ . Its full subcategory of bounded complexes will be denoted by  $K^b(\underline{B})$ . We will write  $M = (M^i)$  if  $M^i$  are the terms of a complex  $M$ ;  $f^i$  will denote the  $i$ th component of a morphism of complexes  $f$ . If we will say that an arrow (or a sequence of arrows) in  $\underline{A}$  yields an object of  $K^b(\underline{B})$ , we will mean by default that the last object of this sequence is in degree 0. We will always extend a “finite”  $\underline{B}$ -complex by 0’s to  $\pm\infty$  (in order to obtain an object of  $K^b(\underline{B}) \subset K(\underline{B})$ ).

We will call a contravariant additive functor  $\underline{\mathcal{C}} \rightarrow \underline{A}$  for an abelian  $\underline{A}$  *cohomological* if it converts distinguished triangles into long exact sequences. For a cohomological  $F$  we will denote  $F \circ [-i]$  by  $F^i$ .

For  $I \in \text{Obj } \underline{\mathcal{C}}$  we will denote the cohomological functor  $\underline{\mathcal{C}}(-, I)$  (from  $\underline{\mathcal{C}}$  into  $\text{Ab}$ ) by  $H_I$ .

On the other hand, we will call a covariant functor  $F$  satisfying this condition a *homological* one or just homology; we denote  $F \circ [i]$  by  $F_i$ . So, for an  $\underline{A}$ -complex  $(M^i, d^i : M^i \rightarrow M^{i+1})$  the object  $\text{Ker}(d^i)/\text{Im } d^{i-1}$  is the  $i$ th homology  $H_i(M)$ . This convention is compatible with the previous papers of the author; yet it forces us to use a somewhat weird numbering for  $H\mathbb{Z}$ -homology of spectra in §2.4.

Let  $\underline{\mathcal{C}}$  be a triangulated category closed with respect to coproducts,  $B \subset \text{Obj } \underline{\mathcal{C}}$ . Then an object  $M$  of  $\underline{\mathcal{C}}$  is called *compact* if the functor  $\underline{\mathcal{C}}(M, -)$  commutes with all small coproducts.

$L$  will always be an arbitrary (fixed) field.  $L\text{-vect}$  will denote the category of finite dimensional  $L$ -vector spaces.

## 1.2 Weight structures: basics

**Definition 1.2.1.** I. A pair of subclasses  $\underline{\mathcal{C}}_{w \leq 0}, \underline{\mathcal{C}}_{w \geq 0} \subset \text{Obj } \underline{\mathcal{C}}$  will be said to define a weight structure  $w$  for a triangulated category  $\underline{\mathcal{C}}$  if they satisfy the following conditions.

(i)  $\underline{\mathcal{C}}_{w \geq 0}, \underline{\mathcal{C}}_{w \leq 0}$  are Karoubi-closed in  $\underline{\mathcal{C}}$  (i.e., contain all  $\underline{\mathcal{C}}$ -retracts of their objects).



(ii) **Semi-invariance with respect to translations.**

$$\underline{C}_{w \leq 0} \subset \underline{C}_{w \leq 0}[1], \underline{C}_{w \geq 0}[1] \subset \underline{C}_{w \geq 0}.$$

(iii) **Orthogonality.**

$$\underline{C}_{w \leq 0} \perp \underline{C}_{w \geq 0}[1].$$

(iv) **Weight decompositions.**

For any  $M \in \text{Obj } \underline{C}$  there exists a distinguished triangle

$$X \rightarrow M \rightarrow Y \rightarrow X[1] \quad (1.2.1)$$

such that  $X \in \underline{C}_{w \leq 0}$ ,  $Y \in \underline{C}_{w \geq 0}[1]$ .

II. The category  $\underline{Hw} \subset \underline{C}$  whose objects are  $\underline{C}_{w=0} = \underline{C}_{w \geq 0} \cap \underline{C}_{w \leq 0}$  and morphisms are  $\underline{Hw}(Z, T) = \underline{C}(Z, T)$  for  $Z, T \in \underline{C}_{w=0}$ , is called the *heart* of  $w$ .

III.  $\underline{C}_{w \geq i}$  (resp.  $\underline{C}_{w \leq i}$ , resp.  $\underline{C}_{w=i}$ ) will denote  $\underline{C}_{w \geq 0}[i]$  (resp.  $\underline{C}_{w \leq 0}[i]$ , resp.  $\underline{C}_{w=0}[i]$ ).

IV. We denote  $\underline{C}_{w \geq i} \cap \underline{C}_{w \leq j}$  by  $\underline{C}_{[i,j]}$  (so it equals  $\{0\}$  if  $i > j$ ).

$\underline{C}^b \subset \underline{C}$  will be the category whose object class is  $\cup_{i,j \in \mathbb{Z}} \underline{C}_{[i,j]}$ .

V. We will say that  $(\underline{C}, w)$  is *bounded* if  $\underline{C}^b = \underline{C}$  (i.e., if  $\cup_{i \in \mathbb{Z}} \underline{C}_{w \leq i} = \text{Obj } \underline{C} = \cup_{i \in \mathbb{Z}} \underline{C}_{w \geq i}$ ).

Respectively, we will call  $\cup_{i \in \mathbb{Z}} \underline{C}_{w \leq i}$  (resp.  $\cup_{i \in \mathbb{Z}} \underline{C}_{w \geq i}$ ) the class of *w-bounded above* (resp. *w-bounded below*) objects; we will say that  $w$  is bounded above (resp. bounded below) if all the objects of  $\underline{C}$  satisfy this property.

VI. Let  $\underline{C}$  and  $\underline{C}'$  be triangulated categories endowed with weight structures  $w$  and  $w'$ , respectively; let  $F : \underline{C} \rightarrow \underline{C}'$  be an exact functor.

$F$  is said to be *weight-exact* (with respect to  $w, w'$ ) if it maps  $\underline{C}_{w \leq 0}$  into  $\underline{C}'_{w' \leq 0}$  and maps  $\underline{C}_{w \geq 0}$  into  $\underline{C}'_{w' \geq 0}$ .

VII. Let  $\underline{B}$  be a full subcategory of a triangulated category  $\underline{C}$ .

We will say that  $\underline{B}$  is *negative* if  $\text{Obj } \underline{B} \perp (\cup_{i > 0} \text{Obj}(\underline{B}[i]))$ .

*Remark 1.2.2.* 1. A simple (though quite useful for this paper) example of a weight structure comes from the stupid filtration on  $K(\underline{B})$  (or on  $K^b(\underline{B})$ ) for an arbitrary additive category  $\underline{B}$ . In this case  $K(\underline{B})_{w \leq 0}$  (resp.  $K(\underline{B})_{w \geq 0}$ ) will be the class of complexes that are homotopy equivalent to complexes concentrated in degrees  $\geq 0$  (resp.  $\leq 0$ ); see Remark 1.2.2(1) of [BoS16].

The heart of this *stupid weight structure* is the Karoubi-closure of  $\underline{B}$  in  $K(\underline{B})$ .

2. A weight decomposition (of any  $M \in \text{Obj } \underline{C}$ ) is (almost) never canonical.

Yet for  $m \in \mathbb{Z}$  we will often need some choice of a weight decomposition of  $M[-m]$  shifted by  $[m]$ . So we obtain a distinguished triangle

$$w_{\leq m}M \rightarrow M \rightarrow w_{\geq m+1}M \quad (1.2.2)$$

with some  $w_{\geq m+1}M \in \underline{C}_{w \geq m+1}$ ,  $w_{\leq m}M \in \underline{C}_{w \leq m}$ ; we will call it an  $m$ -weight decomposition of  $M$ .

We will often use this notation below (though  $w_{\geq m+1}M$  and  $w_{\leq m}M$  are not canonically determined by  $M$ ). Besides, when we will write arrows of the type  $w_{\leq m}M \rightarrow M$  or  $M \rightarrow w_{\geq m+1}M$  we will always assume that they come from some  $m$ -weight decomposition of  $M$ .

3. In the current paper we use the “homological convention” for weight structures; it was previously used in [Wil09], [Heb11], [Bon14], [Wil08], [Wil15b], [Bon15], [Bon13], [Wil15a], [BoI15], and [BoS16] whereas in [Bon10a], [Bon10b], and [BoT15] the “cohomological convention” was used. In the latter convention the roles of  $\underline{C}_{w \leq 0}$  and  $\underline{C}_{w \geq 0}$  are interchanged, i.e., one considers  $\underline{C}^{w \leq 0} = \underline{C}_{w \geq 0}$  and  $\underline{C}^{w \geq 0} = \underline{C}_{w \leq 0}$ . So, a complex  $X \in \text{Obj } K(\underline{A})$  whose only non-zero term is the fifth one (i.e.,  $X^5 \neq 0$ ) has weight  $-5$  in the homological convention, and has weight  $5$  in the cohomological convention. Thus the conventions differ by “signs of weights”;  $K(\underline{A})_{[i,j]}$  is the class of retracts of complexes concentrated in degrees  $[-j, -i]$ .

4. The orthogonality axiom in Definition 1.2.1(I) immediately yields that  $\underline{Hw}$  is negative in  $\underline{C}$ . We will mention a certain converse to this statement below.

Let us recall some basic properties of weight structures. Starting from this moment we will assume that  $\underline{C}$  is (a triangulated category) endowed with a (fixed) weight structure  $w$ .

**Proposition 1.2.3.** *Let  $m \leq l \in \mathbb{Z}$ ,  $M, M' \in \text{Obj } \underline{C}$ ,  $g \in \underline{C}(M, M')$ .*

1. *The axiomatics of weight structures is self-dual, i.e., for  $\underline{D} = \underline{C}^{op}$  (so  $\text{Obj } \underline{D} = \text{Obj } \underline{C}$ ) there exists the (opposite) weight structure  $w^{op}$  for which  $\underline{D}_{w^{op} \leq 0} = \underline{C}_{w \geq 0}$  and  $\underline{D}_{w^{op} \geq 0} = \underline{C}_{w \leq 0}$ .*
2.  *$\underline{C}_{w \geq 0} = (\underline{C}_{w \leq -1})^\perp$  and  $\underline{C}_{w \leq -1} = {}^\perp \underline{C}_{w \geq 1}$ .*
3.  *$\underline{C}_{w \leq 0}$ ,  $\underline{C}_{w \geq 0}$ , and  $\underline{C}_{w=0}$  are additive.*
4. *A direct sum of (a finite collection of)  $m$ -weight decompositions of any  $M_i \in \text{Obj } \underline{C}$  is an  $m$ -weight decomposition of  $\bigoplus M_i$ .*

5. For any (fixed)  $m$ -weight decomposition of  $M$  and an  $l$ -weight decomposition of  $M$  (see Remark 1.2.2(2))  $g$  can be extended to a morphism of the corresponding distinguished triangles:

$$\begin{array}{ccccc}
w_{\leq m}M & \xrightarrow{c} & M & \longrightarrow & w_{\geq m+1}M \\
\downarrow h & & \downarrow g & & \downarrow j \\
w_{\leq l}M' & \longrightarrow & M' & \longrightarrow & w_{\geq l+1}M'
\end{array} \tag{1.2.3}$$

Moreover, if  $m < l$  then this extension is unique (provided that the rows are fixed).

6. Assume that we are given a diagram of the form (1.2.3) and its rows are equal (so,  $M' = M$ ,  $m = l$ ,  $w_{\leq m}M = w_{\leq l}M'$ ); also suppose that  $g = \text{id}_M$  and  $h$  is an idempotent endomorphism, whereas  $\underline{C}$  is Karoubian. Then for the decomposition  $w_{\leq m}M \cong M_1 \oplus M_0$  corresponding to  $h$  (i.e.,  $h$  projects  $w_{\leq m}M$  onto  $M_1$ ) we have  $M_0 \in \underline{C}_{w=m}$ , whereas the upper row of (1.2.3) can be presented as the direct sum of a certain  $m$ -weight decomposition  $M_1 \rightarrow M \rightarrow M_2$  and of the distinguished triangle  $M_0 \rightarrow 0 \rightarrow M_0[1]$ .
7. Assume  $M' \in \underline{C}_{w \geq m}$ . Then any  $g \in \underline{C}(M, M')$  factors through  $w_{\geq m}M$  (for any choice of the latter object).
8. If  $M \in \underline{C}_{w \geq m}$  then  $w_{\leq l}M \in \underline{C}_{[m, l]}$  (for any  $l$ -weight decomposition of  $M$ ). Dually, if  $M \in \underline{C}_{w \leq l}$  then  $w_{\geq m}M \in \underline{C}_{[m, l]}$ .
9. Let  $M \in \underline{C}_{w \leq 0}$  into  $N \in \underline{C}_{w \geq 0}$  and fix some weight decompositions  $X_1[1] \rightarrow M[1] \xrightarrow{f[1]} Y_1[1]$  and  $X_2 \xrightarrow{g} N \rightarrow Y$  of  $M[1]$  and  $N$ , respectively. Then  $Y_1, X_2 \in \underline{C}_{w=0}$  and any morphism from  $M$  into  $N$  can be presented as  $g \circ h \circ f$  for some  $h \in \underline{C}(Y_1, X_2)$ .

*Proof.* Assertions 1, 2, 3, 5, and 8, were proved in [Bon10a] (cf. Remark 1.2.2(4) of [BoS16] and pay attention to Remark 1.2.2(3) above!).

Assertion 4 follows from assertion 3 immediately (since direct sums of distinguished triangles are distinguished).

To prove assertion 6 we note first that  $h$  does yield a certain splitting of  $w_{\leq m}M$  since  $\underline{C}$  is Karoubian. Next, since (1.2.3) is commutative, we obtain that  $c$  factors through  $M_1$ . Hence the upper row of (1.2.3) can be decomposed

into a direct sum of the distinguished triangle  $M_0 \rightarrow 0 \rightarrow M_0[1]$  with a certain triangle  $M_1 \rightarrow M \rightarrow M_2$ . Lastly,  $M_1 \in \underline{C}_{w \leq m}$  and  $M_2 \in \underline{C}_{w \geq m+1}$  (since  $\underline{C}_{w \leq m}$  and  $\underline{C}_{w \geq m+1}$  are Karoubi-closed in  $\underline{C}$ ), whereas  $M_0 \in \underline{C}_{w \leq m} \cap \underline{C}_{w \geq m+1}[-1] = \underline{C}_{w=m}$ .

Assertion 7 follows from assertion 5 immediately.

Lastly, the assumptions of assertion 9 imply that  $Y_1, X_2 \in \underline{C}_{w=0}$  according to assertion 8. The rest of the assertion easily follows from assertion 7 combined with its dual.  $\square$

*Remark 1.2.4.* Diagrams of the form (1.2.3) (also in the case  $l < m$ ) are crucial for this paper.

1. An important type of this diagrams is the one with  $g = id_M$  (for  $M' = M$ ; cf. part 6 of the proposition). Note that for  $m < l$  the corresponding connecting morphisms in (1.2.3) are certainly unique (provided that the rows are fixed); if  $m = l$  then we obtain a certain (non-unique) "modification" of an  $m$ -weight decomposition diagram.

2. One can "compose" diagrams of the form (1.2.3), i.e., for any  $q \in \underline{C}(M', M'')$ ,  $k \in \mathbb{Z}$ , and a morphism of triangles of the form

$$\begin{array}{ccccc} w_{\leq l}M' & \longrightarrow & M' & \longrightarrow & w_{\geq l+1}M' \\ \downarrow & & \downarrow q & & \downarrow \\ w_{\leq k}M'' & \longrightarrow & M'' & \longrightarrow & w_{\geq k+1}M'' \end{array}$$

one can compose its vertical arrows with the ones of (1.2.3) to obtain a morphism of distinguished triangles

$$\begin{array}{ccccc} w_{\leq m}M & \xrightarrow{c} & M & \longrightarrow & w_{\geq m+1}M \\ \downarrow & & \downarrow q \circ g & & \downarrow \\ w_{\leq k}M'' & \longrightarrow & M'' & \longrightarrow & w_{\geq k+1}M'' \end{array}$$

Note that one does not have to assume  $k \geq l$  here (and  $l \geq m$  also is not necessary provided that the existence of (1.2.3) is known in this case). Anyway, if  $k > m$  then the composed diagram obtained this way is the only possible morphism of triangles compatible with  $q \circ g$ .

3. Note also that (1.2.3) can certainly be recovered from its left or right hand square.

### 1.3 On weight complexes

**Definition 1.3.1.** For an object  $M$  of  $\underline{\mathcal{C}}$  (where  $\underline{\mathcal{C}}$  is endowed with a weight structure  $w$ ) choose some  $w_{\leq l}M$  (see Remark 1.2.2(2)) for all  $l \in \mathbb{Z}$ . For all  $l \in \mathbb{Z}$  connect  $w_{\leq l-1}M$  with  $w_{\leq l}M$  using Proposition 1.2.3(5) (i.e., we consider those unique connecting morphisms that are compatible with  $id_M$ ; see Remark 1.2.4(1)). Next, take the corresponding triangles

$$w_{\leq l-1}M \rightarrow w_{\leq l}M \rightarrow M^{-l}[l] \quad (1.3.1)$$

(so, we just introduce the notation for the corresponding cones). All of these triangles together with the corresponding morphisms  $w_{\leq l}M \rightarrow M$  are called a choice of a *weight Postnikov tower* for  $M$ , whereas the objects  $M^i$  together with the morphisms connecting them (obtained by composing the morphisms  $M^{-l} \rightarrow (w_{\leq l-1}M)[1-l] \rightarrow M^{-l+1}$  that come from two consecutive triangles of the type (1.3.1)) will be denoted by  $t(M)$  and is said to be a choice of a *weight complex* for  $M$ .

Respectively, for some  $M, M' \in \text{Obj } \underline{\mathcal{C}}$ ,  $g \in \underline{\mathcal{C}}(M, M')$ , and some choices of their weight complexes we will say that a collection of arrows between the terms of these complexes is a choice of  $t(g)$  whenever these arrows come from some morphism of the corresponding weight Postnikov towers that is compatible with  $g$ .

Let us recall some basic properties of weight complexes.

**Proposition 1.3.2.** *Let  $M, M' \in \text{Obj } \underline{\mathcal{C}}$ ,  $g \in \underline{\mathcal{C}}(M, M')$  (where  $\underline{\mathcal{C}}$  is endowed with a weight structure  $w$ ).*

*Then the following statements are valid.*

1. *Any choice of  $t(M) = (M^i)$  is a complex indeed (i.e., the square of the boundary is zero); all  $M^i$  belong to  $\underline{\mathcal{C}}_{w=0}$ .*
2. *Any choice of  $t(g)$  is a  $C(\underline{Hw})$ -morphism from the corresponding  $t(M)$  to  $t(M')$ .*
3.  *$M$  determines its weight complex  $t(M)$  up to homotopy equivalence. In particular, if  $M \in \underline{\mathcal{C}}_{w \geq 0}$ , then any choice of  $t(M)$  is  $K(\underline{Hw})$ -isomorphic to a complex with non-zero terms in non-positive degrees only; if  $M \in \underline{\mathcal{C}}_{w \leq 0}$  then  $t(M)$  is isomorphic to a complex with non-zero terms in non-negative degrees only.*

4.  $g$  determines its weight complex  $t(g)$  up to the following weak homotopy equivalence relation: for  $\underline{Hw}$ -complexes  $A, B$  and morphisms  $m_1, m_2 \in C(\underline{Hw})(A, B)$  we write  $m_1 \sim m_2$  if  $m_1 - m_2 = d_B h + j d_A$  for some collections of arrows  $j^*, h^* : A^* \rightarrow B^{*-1}$ .

Besides, this equivalence relation is respected by compositions, and so considering morphisms in  $K(\underline{Hw})$  modulo this relation we obtain an additive category  $K_w(\underline{Hw})$ .

5. There exist choices of  $t(M)$ ,  $t(M')$ , and (a compatible choice of)  $t(g)$  such that the cone of  $t(g)$  is a choice of a weight complex of  $\text{Cone}(g)$ .
6. Let  $\underline{C}'$  be a triangulated category endowed with a weight structure  $w'$ ; let  $F : \underline{C} \rightarrow \underline{C}'$  be a weight-exact functor. Then for any choice of  $t(M)$  (resp. of  $t(g)$ ) the complex  $F(M^i)$  (resp. of  $F_*(t(g))$ ) yields a weight complex of  $F(M)$  (resp. a choice of  $t(F(g))$ ) with respect to  $w'$ .

*Proof.* Assertions 1–5 follow immediately from Theorem 3.2.2(II) and Theorem 3.3.1 of [Bon10a].

Assertion 6 is an immediate consequence of the definition of a weight complex (and of weight-exact functors).  $\square$

*Remark 1.3.3.* 1. The term "weight complex" originates from [GiS96], where a certain complex of Chow motives was constructed for a variety  $X$  over a characteristic 0 field. The weight complex functor of Gillet and Soulé can be obtained via applying the ("triangulated motivic") weight complex functor  $DM_{gm}^{eff} \rightarrow K^b(\text{Chow}^{eff})$  (or  $DM_{gm} \rightarrow K^b(\text{Chow})$ ) to the *motif with compact support* of  $X$  (see Proposition 6.6.2 of [Bon09]). Certainly, our notion of a weight complex functor is much more general.

2. The weak homotopy equivalence relation was introduced in §3.1 of [Bon10a] (independently from the earlier Theorem 2.1 of [Bar05]). It has several nice properties; in particular, the identity of a complex if weakly homotopic to 0 if and only if this complex is contractible (see Proposition 3.1.8(1) of [Bon10a]).
3. Let  $\underline{B}$  be an additive category and  $k \leq l \in (\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\})$ ; we assume in addition that  $(k, l)$  differs both from  $(-\infty, -\infty)$  and from  $(+\infty, +\infty)$ . For  $m_1, m_2 : A \rightarrow B$  (for  $A, B \in \text{Obj } C(\underline{B})$ ) we will write  $m_1 \sim_{[k, l]} m_2$  if  $m_1 - m_2$  is weakly homotopic to certain  $m_0$  such that

$m_0^i = 0$  for  $k \leq i \leq l$  (and  $i \in \mathbb{Z}$ ). In particular,  $m_1$  is weakly equivalent to  $m_2$  if  $m_1 \smile_{[-\infty, +\infty]} m_2$ .

Then for  $m : A \rightarrow B$  we have  $m \smile_{[k, l]} 0$  if there exist two sequences  $h^j, g^j \in \underline{B}(A^j, B^{j-1})$  (for  $j \in \mathbb{Z}$ ) such that  $d_B^{j-1} \circ h^j + g^{j+1} \circ d_A^j = m^j$  for  $k \leq j \leq l$  and  $d_B^j \circ h^j \circ d_A^j = d_B^j \circ g^j \circ d_A^j$  for all  $j \in \mathbb{Z}$ . Hence in the case  $k = l$  we have  $m \smile_{[k, l]} 0$  if and only if  $m$  is homotopic to an  $m_0$  such that  $m_0^k = 0$  (since we can take  $m_0 = m + d_B \circ f + f \circ d_B$ , where  $f^i = h^i$  for  $i \leq k$  and  $f^i = g^i$  for  $i > k$ ). Besides, (in contrast to the homotopy equivalence relation for morphism) the weak homotopy possesses the following property:  $m \smile_{[k, l]} 0$  if and only if  $m \smile_{[i, i]} 0$  for all  $k \leq i \leq l$ . So, the weak homotopy relation has certain advantages over (the usual) homotopy one.

Besides, if  $\underline{B} \subset \underline{B}'$  and  $m_1 \smile_{[k, l]} m_2$  in  $K(\underline{B}')$  (in the notation introduced above) then  $m_1 \smile_{[k, l]} m_2$  in  $K(\underline{B})$  also. Lastly (according to the previous part of this remark), if  $id_A$  is weakly homotopic to zero then  $A$  is contractible.

4. Combining these observations with Proposition 3.2.4(2) of [Bon10a] one can easily deduce the following statement.

Adopt the notation of Proposition 1.3.2 and fix certain choices of  $t(M)$  and  $t(M')$  (as well of the of the corresponding analogues of the triangles (1.3.1)). Then the possible choices of  $t(f)$  corresponding to this data form a (whole) equivalence class in  $C(\underline{Hw})(t(M), t(M'))$  with respect to the weak homotopy relation.

5. Below we will need the following properties of  $\underline{B}$ -complexes: for  $A \in \text{Obj } K(\underline{B})$  we will write  $A \smile_w 0$  (resp.  $A \smile^w 0$ ) if  $A$  is homotopy equivalent to a complex concentrated in non-positive (resp. in non-negative) degrees.

Certainly, if  $A \smile_w 0$  then  $id_A \smile_{[1, +\infty]} 0_A$ ; if  $A \smile^w 0$  then  $id_A \smile_{[-\infty, -1]} 0_A$ . Now we prove that the converse implications are valid also; it certainly suffices to verify the first of these statements (since the second one is its dual).

If  $id_A \smile_{[1, +\infty]} 0_A$  then  $id_A$  is weakly equivalent to  $g \in K(\underline{B})(A, A)$  such that  $g^i = 0$  for all  $i > 0$ . Then  $g$  is an automorphism of  $A$  (see Proposition 3.1.8(1) of [Bon10a]) and it can be factored through the stupid truncation morphism  $A \rightarrow A^{\leq 0}$  (for  $A^{\leq 0} = \cdots \rightarrow A^{-1} \rightarrow$

$A^0 \rightarrow 0 \rightarrow 0 \dots$ ). Hence  $A$  is a  $K(\underline{B})$ -retract of  $A^{\leq 0}$ . Lastly, Theorem 3.1 of [Sch11] (cf. also Proposition 4.2.4 of [Sos15]; one should take  $F(-) = \coprod_{i \geq 0} -[2i]$  in it) yields that  $A$  is homotopy equivalent to a complex concentrated in non-positive degrees indeed.

6. Our definition of weight complexes is not (quite) self-dual, since for describing the weight complex of  $M \in \text{Obj } \underline{\mathcal{C}}$  in  $\underline{\mathcal{C}}^{op}$  (with respect to  $w^{op}$ ; see Proposition 1.2.3(1)) we have to consider  $w_{\geq i} M$  instead. One may say that there exist "right" and "left" weight complex functors possessing similar properties. They are actually isomorphic if  $\underline{\mathcal{C}}$  embeds into a category that possesses a model (see Remark 1.5.9(1) of [Bon10a]); the author does not know whether this is true (for weight complexes of morphisms) in general. Yet Proposition 2.3.1(3) below (along with Remark 2.1.3) demonstrate that switching to left weight complexes would not have affected the relation  $\sim_{[k,l]}$  for weight complexes of morphisms.
7. In "most of" the known examples of weight structures of weight structures  $\underline{\mathcal{C}}$  either admits a differential graded enhancement (in the sense introduced in [BoK90]; see §6 of [Bon10a]) or  $\underline{\mathcal{C}} = SH$  (see §2.4 below; the only notable exception known to the author is the Gersten weight structure for a certain category of *motivic prospectra* constructed in [Bon13]). In both of these cases  $t$  can be "enhanced" to an exact functor  $t_{st} : \underline{\mathcal{C}} \rightarrow K(\underline{Hw})$ .

## 1.4 Weight filtrations, virtual $t$ -truncations, and adjacient structures

Now suppose that we are given a cohomological (or just any contravariant) functor  $H : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{A}}$ , where  $\underline{\mathcal{A}}$  is an abelian category. We recall that weight structures yield functorial *weight filtrations* for  $H(-)$  (that vastly generalize the weight filtration of Deligne for étale and singular homology of varieties). We also consider virtual  $t$ -truncations for  $H$  (as defined in §2.5 of [Bon10a] and studied in more detail in §2 of [Bon10b]). The latter allow us to "slice"  $H$  into  $w$ -pure pieces (cf. Remark 3.2.2(1) below); we will only formulate a few of their properties here. These truncations behave as if they were given by truncations of  $H$  in some triangulated 'category of functors'  $\underline{\mathcal{D}}$  with respect to some  $t$ -structure (whence the name). Moreover, this is often actually the case (in particular, in the "motivic" and "topological" settings



that will be discussed below); yet the definition does not require the existence of  $\underline{D}$  (and so, does not depend on its choice). Our choice of the numbering for them is motivated by the cohomological convention for  $t$ -structures (that we use in this paper following [BBD82]); this convention combined with the homological numbering for weight structures causes certain (somewhat weird) – signs in the definitions of this section.

**Definition 1.4.1.** Let  $H : \underline{C} \rightarrow \underline{A}$  be a contravariant functor,  $m \leq n \in \mathbb{Z}$ .

1. We define the weight filtration for  $H(M)$  as

$$W^m(H)(X) = \text{Im}(H(w_{\geq m}M) \rightarrow H(M));$$

here we take an arbitrary choice of  $w_{\geq m}M$ .

2. We define the functor  $\tau^{\geq -n}(H)$  by the correspondence

$$M \mapsto \text{Im}(H(w_{\leq n+1}M) \rightarrow H(w_{\leq n}M));$$

here we take arbitrary choices of  $w_{\leq n}M$  and  $w_{\leq n+1}M$ , and connect them as in Remark 1.2.4(1).

3. If  $H$  is cohomological, we will say that it is of weight range  $\geq m$  if it annihilates  $\underline{C}_{w \leq m-1}$ ; we will say that it is of weight range  $[m, n]$  if it also annihilates  $\underline{C}_{w \geq n+1}$ .

We list some of the properties of this notions; none of them are really new.

**Proposition 1.4.2.** *In the notation of the previous definition the following statements are valid.*

1.  $W^m H(M)$  and  $\tau^{\geq m}(H)$  are  $\underline{C}$ -functorial in  $M$  (for any  $m$ ; in particular, they do not depend on the choices of the corresponding weight decompositions of  $M$ ).
2. If  $H$  is cohomological then  $\tau^{\geq m}(H)$  also is.
3. The functor  $\underline{C}(-, M) : \underline{C} \rightarrow \text{Ab}$  is of weight range  $\geq m$  if and only if  $M \in \underline{C}_{w \geq m}$ .
4. If  $H$  is (cohomological and) of weight range  $\geq m$  then  $\tau^{\geq -n}(H)$  is (also cohomological and) of weight range  $[m, n]$ .

5. If  $H$  is of weight range  $[m, n]$  then the morphism  $H(w_{\geq m}M) \rightarrow H(M)$  is surjective and the morphism  $H(M) \rightarrow H(w_{\leq n}M)$  is injective (here we take arbitrary choice of the corresponding weight decompositions of  $M$  and apply  $H$  to their connecting morphisms).

*Proof.* The first part of assertion 1 is given by Proposition 2.1.2(2) of [Bon10a].

Its second part along with assertion 2 is contained in Theorem 2.3.1 of [Bon10b].

Assertions 3,4 are immediate from Proposition 1.2.3(2,8).

Assertion 5 is an immediate consequence of assertion 2 (consider the corresponding long exact sequences for  $H$ -cohomology).  $\square$

Now we would like to relate virtual  $t$ -truncations to actual ones. We will give the definition of a  $t$ -structure here mostly for fixing the notation; next we define adjacent weight and  $t$ -structures.

**Definition 1.4.3.** I. A pair of subclasses  $\underline{C}^{t \geq 0}, \underline{C}^{t \leq 0} \subset \text{Obj } \underline{C}$  will be said to define a  $t$ -structure  $t$  if they satisfy the following conditions:

- (i)  $\underline{C}^{t \geq 0}, \underline{C}^{t \leq 0}$  are strict, i.e., contain all objects of  $\underline{C}$  isomorphic to their elements.
- (ii)  $\underline{C}^{t \geq 0} \subset \underline{C}^{t \geq 0}[1], \underline{C}^{t \leq 0}[1] \subset \underline{C}^{t \leq 0}$ .
- (iii)  $\underline{C}^{t \leq 0}[1] \perp \underline{C}^{t \geq 0}$ .
- (iv) For any  $M \in \text{Obj } \underline{C}$  there exists a  $t$ -decomposition distinguished triangle

$$A \rightarrow M \rightarrow B \rightarrow A[1] \quad (1.4.1)$$

such that  $A \in \underline{C}^{t \leq 0}, B \in \underline{C}^{t \geq 0}[-1]$ .

II For  $m \in n \in \mathbb{Z}$  we will denote  $\underline{C}^{t \geq 0}[n]$  by  $\underline{C}^{t \geq -n}$ ; we also consider  $\underline{C}^{t \leq -m} = \underline{C}^{t \leq 0}[m]$  and  $\underline{C}[m, n] = \underline{C}^{t \leq -m} \cap \underline{C}^{t \geq -n}$  (cf. Definition 1.2.1(IV); the  $SH$ -version of this notation can be found in §3.2 of [Mar83]).

III If  $w$  is a weight structure for  $\underline{C}$  and  $t$  is a  $t$ -structure for it we will say that  $w$  is (left) adjacent to  $t$  (or that  $t$  is adjacent to  $w$ ) if  $\underline{C}^{t \leq 0} = \underline{C}_{w \geq 0}$ .

We recall a few well-known properties of  $t$ -structures and some basics on adjacent structures.

*Remark 1.4.4.* Let  $m \leq n \in \mathbb{Z}$ ; let  $t$  be a  $t$ -structure for  $\underline{C}$ .

1. Recall that the triangle (1.4.1) is canonically (and functorially) determined by  $M$ . So for  $A'[-1-n] \rightarrow M[-1-n] \rightarrow B'[-1-n]$  being a  $t$ -decomposition of  $M[-1-n]$  we can denote  $B'$  by  $t^{\leq -n}M$  (and this notation

is functorial in contrast to the setting of weight structures). If  $M \in \underline{C}^{t \leq -m}$  then we (certainly) have  $t^{\leq -n} M \in \underline{C}[m, n]$ .

Recall also that  $\underline{C}^{t \leq -m} = (\underline{C}^{t \geq 1-m})^\perp$ .

2. The heart of  $t$  is defined similarly to that of  $w$ : this is a full subcategory  $\underline{Ht}$  of  $\underline{C}$  with  $\text{Obj } \underline{Ht} = \underline{C}[0, 0]$ . Recall that  $\underline{Ht}$  is necessarily an abelian category with short exact sequences corresponding to distinguished triangles in  $\underline{C}$ .

3. The following statement is a particular case of Proposition 2.5.4(1) of [Bon10b] (cf. also Proposition 2.5.6(1) of *ibid.*): for  $H = \underline{C}(-, M)$  we have  $\tau^{\geq -n}(H) \cong \underline{C}(-, t^{\leq -n} M)$ .

4. We will mostly be interested in "topological" and motivic examples of weight structures. In all of these examples (see §4.6 of [Bon10a] or §2.4 below for the  $SH$ -one, whereas the most general Chow weight structures for motives over various base schemes are discussed in [BoI15]; cf. also Remark 2.2.4(3) below)  $\underline{C}$  is closed with respect to small coproducts and there exists a small additive negative (see Definition 1.2.1(VII)) subcategory  $\underline{B} \subset \underline{C}$  such that  $\underline{B}$  weakly generates  $\underline{C}$  and the objects of  $\underline{B}$  are compact in  $\underline{C}$ .

In this setting adjacent  $w$  and  $t$  for  $\underline{C}$  can be constructed as follows: one should take  $\underline{C}_{w \geq 0} = \underline{C}^{t \leq 0} = (\cup_{i < 0} \text{Obj } \underline{B}[i])^\perp$  and recover the "remaining halves of these structures" using the corresponding orthogonality conditions (see Theorem 4.5.2 of [Bon10a] or §A.1 of [Bon14]). Besides in this setting  $\underline{Hw}$  is the idempotent completion of the category of all small coproducts of objects of  $\underline{B}$ , whereas  $\underline{C}[0, 0] = (\cup_{i \in \mathbb{Z} \setminus \{0\}} \text{Obj } \underline{B}[i])^\perp$ . Furthermore, the correspondence sending  $M \in \text{Obj } \underline{C}$  into the functor  $\underline{B}^{op} \rightarrow Ab : B \mapsto \underline{C}(B, M)$ , yields an exact equivalence of  $\underline{Ht}$  with the (abelian) category of all additive functors  $\underline{B}^{op} \rightarrow Ab$ .

5. Somewhat more simple (yet certainly not "trivial") examples of adjacent structures are given by appropriate versions of  $D(\underline{A})$  for  $\underline{A}$  being an abelian category with enough projectives; then we have  $\underline{Ht} \cong \underline{A}$  and  $\underline{Hw} \cong \text{Proj } \underline{A}$  (the corresponding  $w$  is obtained by considering projective hyperresolutions of  $\underline{A}$ -complexes).

6. A full (and functorial) description of functors of weight range  $[m, m]$  is immediate from Corollary 2.3.4 of [Bon13]; we will give it in Remark 3.2.2 below.

7. One can certainly define right adjacent structures, whereas weight filtrations and virtual  $t$ -truncations can easily be defined for homology (by a simple reversion of arrows; cf. §2 of [Bon10a]).

We are (currently) not interested in the corresponding versions of our

statements since we do not have right adjacent  $t$ -structures in the cases interesting to us.

## 2 On morphisms killing weights and the relation to weight complexes

Recall that (a fixed triangulated category)  $\underline{\mathcal{C}}$  is assumed to be endowed with a weight structure  $w$ .

In §2.1 we define morphisms killing weights  $m, \dots, n$  and objects without these weights; we give several equivalent definitions of these notions.

In §2.2 we establish several interesting properties of our notions. In particular, we prove that an object without weights  $m, \dots, n$  admits a (weight) *decomposition avoiding these weights* (in the sense defined by Wildeshaus) if  $\underline{\mathcal{C}}$  is Karoubian. We also relate killing weights to weight filtrations for cohomology and to virtual  $t$ -truncations of certain representable functors; so we obtain certain "cohomological detectors" for killing weights (and being without weights  $m, \dots, n$  for objects).

In §2.3 we relate our main notions to the weight complex functor  $t$ . In particular,  $M$  is without weights  $m, \dots, n$  if and only if  $t(M)$  possesses this property. Next we prove that  $t$  is "conservative up to degenerate cones" (significantly improving the corresponding results of §3 of [Bon10a]).

In §2.4 we apply our result to the study of the (topological) stable homotopy category  $SH$  (endowed with the spherical weight structure that was defined in §4.6 of [Bon10a]). We fill in some gaps in the arguments of *ibid.* and also explain what our (new) definitions and results mean in this setting (they are closely related to the homology and cohomology given by Eilenberg-MacLane spectra).

### 2.1 Morphisms that kill certain weights: equivalent definitions

**Proposition 2.1.1.** *Let  $g \in \underline{\mathcal{C}}(M, N)$  (for some  $M, N \in \text{Obj } \underline{\mathcal{C}}$ );  $m \leq n \in \mathbb{Z}$ . Then the following conditions are equivalent.*

1. *There exists a choice of  $w_{\leq n}M$  and  $w_{\geq m}N$  such that the composed morphism  $w_{\leq n}M \rightarrow M \xrightarrow{g} N \rightarrow w_{\geq m}N$  is zero (here the first and*

the third morphism in this chain come from the corresponding weight decompositions).

2. There exists a choice of  $w_{\leq n}M$  and  $w_{\leq m-1}N$  and of a morphism  $h$  making the square

$$\begin{array}{ccc} w_{\leq n}M & \longrightarrow & M \\ \downarrow h & & \downarrow g \\ w_{\leq m-1}N & \longrightarrow & N \end{array} \quad (2.1.1)$$

commutative.

3. There exists a choice of  $w_{\geq n+1}M$  and  $w_{\geq m}N$  and of a morphism  $j$  making the square

$$\begin{array}{ccc} M & \longrightarrow & w_{\geq n+1}M \\ \downarrow g & & \downarrow j \\ N & \longrightarrow & w_{\geq m}N \end{array} \quad (2.1.2)$$

commutative.

4. Any choice of an  $n$ -weight decomposition of  $M$  and an  $m-1$ -weight decomposition of  $N$  can be completed to a morphism of distinguished triangles of the form

$$\begin{array}{ccccccc} w_{\leq n}M & \longrightarrow & M & \longrightarrow & w_{\geq n+1}M \\ \downarrow h & & \downarrow g & & \downarrow j \\ w_{\leq m-1}N & \longrightarrow & N & \longrightarrow & w_{\geq m}N \end{array} \quad (2.1.3)$$

5. For any choice of  $m-1$ - and  $n$ -weight decompositions of  $M$  and  $N$ , and for  $a$  and  $b$  being the corresponding (canonical) connecting morphisms  $w_{\leq m-1}M \rightarrow w_{\leq n}M$  and  $w_{\leq m-1}N \rightarrow w_{\leq n}N$  respectively (see Remark 1.2.4(1)), there exists a commutative diagram

$$\begin{array}{ccccccc} w_{\leq m-1}M & \xrightarrow{a} & w_{\leq n}M & \longrightarrow & M \\ \downarrow c & & \downarrow d & & \downarrow g \\ w_{\leq m-1}N & \xrightarrow{b} & w_{\leq n}N & \longrightarrow & N \end{array} \quad (2.1.4)$$

along with a morphism  $h \in \underline{C}(w_{\leq n}M, w_{\leq m-1}N)$  that turns the corresponding "halves" of the left hand square of (2.1.4) into commutative triangles.

6. For any choice of the diagram (2.1.4) as above its left hand commutative square can be completed to a morphism of triangles as follows:

$$\begin{array}{ccccc}
w_{\leq m-1}M & \xrightarrow{a} & w_{\leq n}M & \longrightarrow & \text{Cone}(a) \\
\downarrow c & & \downarrow d & & \downarrow 0 \\
w_{\leq m-1}N & \xrightarrow{b} & w_{\leq n}N & \longrightarrow & \text{Cone}(b)
\end{array} \tag{2.1.5}$$

7. There exists some choice of (2.1.4) such that the corresponding diagram (2.1.5) is a morphism of triangles.

*Proof.* Conditions 1, 2, and 3 are equivalent by Proposition 1.1.9 of [BBD82] (that is easy; in particular, the long exact sequence  $\cdots \rightarrow \underline{C}(w_{\leq n}M, w_{\leq m-1}N) \rightarrow \underline{C}(w_{\leq n}M, N) \rightarrow \underline{C}(w_{\leq n}M, w_{\geq m}N) \cdots$  yields that condition 1 is equivalent to 2).

Loc. cit. also yields that any of these conditions implies the existence of some diagram of the form (2.1.3) for the corresponding choices of rows. One also obtains a diagram of this form for arbitrary choices of these weight decompositions by composing this diagram with the corresponding "change of weight decompositions" diagrams (see Remark 1.2.4(1,2)); so we obtain condition 4. On the other hand, the latter condition obviously implies conditions 1, 2, and 3.

Next, condition 5 certainly implies condition 2. Conversely, in order to obtain the commutative diagrams in condition 5 it suffices to take  $a$  and  $b$  being the canonical connecting morphisms  $w_{\leq m-1}M \rightarrow w_{\leq n}M$  and  $w_{\leq m-1}N \rightarrow w_{\leq n}N$  (see Remark 1.2.4(1)),  $c = h \circ a$ , and  $d = b \circ h$ .

Next, condition 6 certainly yields condition 7. Now, consider the long exact sequence  $\cdots \rightarrow \underline{C}(w_{\leq n}M, w_{\leq m-1}N) \rightarrow \underline{C}(w_{\leq n}M, w_{\leq n}N) \rightarrow \underline{C}(w_{\leq n}M, \text{Cone } b) \rightarrow \cdots$  (for an arbitrary choice of (2.1.4)). If condition 7 is fulfilled, the composed morphism  $w_{\leq n}M \xrightarrow{d} w_{\leq n}N \rightarrow \text{Cone } b$  is zero; hence there exists an  $h \in \underline{C}(w_{\leq n}M, w_{\leq m-1}N)$  making the corresponding triangle (a "half" of the left hand square in (2.1.5)) commutative. Combining this with the commutativity of the right hand square in (2.1.4) we obtain condition 2 once more.

It remains to verify that condition 5 implies condition 6. The aforementioned long exact sequence yields the vanishing of the corresponding composed morphism  $w_{\leq n}M \rightarrow \text{Cone } b$ , whereas the long exact sequence  $\cdots \rightarrow \underline{C}(w_{\leq n}M, w_{\leq m-1}N) \rightarrow \underline{C}(w_{\leq m-1}M, w_{\leq m-1}N) \rightarrow \underline{C}(\text{Cone}(a)[-1], w_{\leq m-1}N) \rightarrow$

... yields the vanishing of the composed morphism  $\text{Cone}(a)[-1] \rightarrow w_{\leq m-1}N$ . We obtain that (2.1.5) is a morphism of triangles indeed.  $\square$

Now we give the main definitions of this paper.

**Definition 2.1.2.** 1. We will say that a morphism  $g$  *kills weights*  $m, \dots, n$  if it satisfies the equivalent conditions of the previous proposition (and we will say that  $f$  kills weight  $m$  if  $m = n$ ). We will denote the class of all  $\underline{C}$ -morphisms killing weights  $m, \dots, n$  by  $\text{Mor}_{[m, n]} \underline{C}$ .

2. We will say that an object  $M$  is *without weights*  $m, \dots, n$  if  $\text{id}_M$  kills weights  $m, \dots, n$ . We will denote the class of  $\underline{C}$ -objects without weights  $m, \dots, n$  by  $\underline{C}_{w \notin [m, n]}$ .

*Remark 2.1.3.* 1. Obviously, these definitions are self-dual in the following natural sense:  $g \in \text{Mor}_{[m, n]} \underline{C}$  (resp.  $M \in \underline{C}_{w \notin [m, n]}$ ) if and only if  $g$  kills  $w^{op}$ -weights  $-n, \dots, -m$  (resp.  $M$  is without  $w^{op}$ -weights  $-n, \dots, -m$ ) in  $\underline{D} = \underline{C}^{op}$  (see Proposition 1.2.3(1)).

2. Now we describe a simple example that illustrates our definitions.

Let  $\underline{B} = L - \text{vect}$  (more generally, one can consider any semi-simple abelian category here); we endow  $\underline{C} = K(\underline{B})$  or  $\underline{C} = K^b(\underline{B})$  with the stupid weight structure  $w$  (see Remark 1.2.2(1)). Then  $M \in \underline{C}_{w \leq 0}$  (resp.  $\in \underline{C}_{w \geq 0}$ ) if and only if the homology  $H_i(M) = H_0(M[i])$  (see the convention in §1.1) vanishes for  $i < 0$  (resp. for  $i > 0$ ). Hence  $g \in \underline{C}(M, N)$  kills weights  $m, \dots, n$  (resp.  $M$  is without weights  $m, \dots, n$ ) if and only if we have  $\underline{C}(-, K[i])(g) = 0$  (resp.  $\underline{C}(M, K[i]) = 0$ ; so we put  $K$  in degree  $-i$ ) for all  $i \in \mathbb{Z}$ ,  $m \leq i \leq n$ . Thus the functors  $\underline{C}(-, K[i])$  for  $m \leq i \leq n$  yield a collection of cohomology theories that detect whether  $g \in \text{Mor}_{[m, n]} \underline{C}$  and  $M \in \underline{C}_{w \notin [m, n]}$ . We do not have so simple "detecting families" of functors in general; yet we will construct quite interesting detecting classes of cohomology below (see Theorem 2.2.3 for the general case and Proposition 2.4.3(2) for the case  $\underline{C} = SH$ ). We prefer considering cohomological detectors (in this paper) for the reasons explained in Remark 1.4.4(4) (cf. also Remark 2.2.4(6) below).

## 2.2 Some properties of our main notions

**Theorem 2.2.1.** *Let  $M, N, O \in \text{Obj } \underline{C}$ ,  $h \in \underline{C}(N, O)$ , and assume that a morphism  $g \in \underline{C}(M, N)$  kills weights  $m, \dots, n$  for some  $m \leq n \in \mathbb{Z}$ . Then the following statements are valid.*

1. Assume  $m \leq m' \leq n' \leq n$ . Then  $g$  also kills weights  $m', \dots, n'$ .
2.  $\text{Mor}_{[m, n]} \underline{C}$  is closed with respect to direct sums and retracts (i.e.,  $\bigoplus g_i$  kills weights  $m, \dots, n$  if and only if all  $g_i$  do that).
3. Assume that  $h$  kills weights  $m', \dots, m-1$  for some  $m' < m$ . Then  $h \circ g$  kills weights  $m', \dots, n$ .
4. Let  $F : \underline{C} \rightarrow \underline{D}$  be a weight-exact functor (with respect to a certain weight structure for  $\underline{D}$ ) and assume that  $h$  kills weights  $m, \dots, n$ . Then  $F(h)$  kills these weights also.
5. For  $F$  and  $h$  as in the previous assertion assume that  $F$  is a full embedding and  $F(h) \in \text{Mor}_{[m, n]} \underline{D}$ . Then  $h \in \text{Mor}_{[m, n]} \underline{C}$ .
6. Assume that  $O$  is without weights  $m, \dots, n$  as well as without weights  $n+1, \dots, n'$  for some  $n' > n$ . Then  $O \in \underline{C}_{w \notin [m', n]}$ .
7. Let there exist a distinguished triangle

$$X \rightarrow O \rightarrow Y \tag{2.2.1}$$

with  $X \in \underline{C}_{w \leq m-1}$ ,  $Y \in \underline{C}_{w \geq n+1}$  (we call it a decomposition avoiding weights  $m, \dots, n$  for  $M$ ). Then (2.2.1) yields  $l$ -weight decompositions of  $O$  for any  $l \in \mathbb{Z}$ ,  $m-1 \leq l \leq n$ . Besides,  $O$  is without weights  $m, \dots, n$ , and this triangle is unique up to a canonical isomorphism.

8. Assume that  $\underline{C}$  is Karoubian. Then the converse to the previous assertion is true also (i.e., any  $O$  without weights  $m, \dots, n$  admits a decomposition avoiding weights  $m, \dots, n$ ).

*Proof.* 1. Easy (if we use condition 2 of Proposition 2.1.1) if we apply Remark 1.2.4(1,2) (once more).

2. Easy (if we use conditions 1 and 4 of Proposition 2.1.1); recall Proposition 1.2.3(4).

3. Easy since we can compose the diagrams given by Proposition 2.1.1(2); see Remark 1.2.4(2) again.

4. Applying  $F$  to (some choice of) the vanishing for  $h$  given by condition 1 of Proposition 2.1.1 we get this condition for  $F(h)$ .



5. For any choice of  $w_{\leq n}M$  and  $w_{\geq m}N$  the composed morphism  $F(w_{\leq n}M) \rightarrow F(M) \xrightarrow{g} N \rightarrow F(w_{\geq m}N)$  is zero (see condition 2 of Proposition 2.1.1); hence this condition is fulfilled for  $h$ .
6. Immediate from the previous assertion (since  $id_O \circ id_O = id_O$ ).
7. Each statement in this assertion easily follows from the previous ones. (2.2.1) yields the corresponding  $l$ -weight decompositions of  $O$  just by definition. We obtain that  $O$  is without weights  $m, \dots, n$  immediately (here we can use either condition 2 or condition 3 of Proposition 2.1.1). This triangle (2.2.1) is canonical by Proposition 1.2.3(5) (if we take  $M = M' = O$ ,  $g = id_O$ ,  $m = n - 1$  and  $l = n$  in it).
8. The idea is to "modify" any (fixed)  $n$ -decomposition of  $O$  using Proposition 1.2.3(6).

We also fix an  $m$ -weight decomposition of  $O$ . By Proposition 2.1.1(2) there exists a commutative square

$$\begin{array}{ccc} w_{\leq n}O & \longrightarrow & O \\ \downarrow z & & \downarrow id_O \\ w_{\leq m-1}O & \longrightarrow & O \end{array}$$

Next, Proposition 1.2.3 yields the existence and uniqueness of the square

$$\begin{array}{ccc} w_{\leq m-1}O & \longrightarrow & O \\ \downarrow t & & \downarrow id_O \\ w_{\leq n}O & \longrightarrow & O \end{array}$$

Now, we can consider multiple compositions of these squares (see Remark 1.2.4). Hence the aforementioned uniqueness statement yields  $t = t \circ z \circ t$ . Thus the morphism  $u = t \circ z$  is idempotent, and the square

$$\begin{array}{ccc} w_{\leq n}O & \longrightarrow & O \\ \downarrow u & & \downarrow id_O \\ w_{\leq n}O & \longrightarrow & O \end{array}$$

is commutative. Now we apply Proposition 1.2.3(6); for  $X$  being the "image" of  $u$  we obtain an  $n$ -weight decomposition  $X \rightarrow O \rightarrow Y$ . It remains to note that  $X \in \underline{C}_{w_{\leq m-1}}$  since  $u$  factors through  $w_{\leq m-1}O$ .

□

*Remark 2.2.2.* 1. The existence of a decomposition of  $O$  avoiding weights  $m, \dots, n$  means precisely that  $O$  is without weights  $m, \dots, n$  in the sense of Definition 1.10 of [Wil09]. So, our definition of this notion is equivalent to the (older) definition of Wildeshaus (who introduced this term) if  $\underline{C}$  is Karoubian (still cf. the example in §3.3.2 below). So, the uniqueness statement in Theorem 2.2.1(7) is a particular case of the functoriality result given by Corollary 1.9 of [Wil09].

2. Certainly,  $\underline{C}_{w \notin [m, n]}$  is closed with respect to retracts and (finite) direct sums (use part 2) of the theorem; a direct proof is easy also).
3. Certainly, parts 1–3 of Theorem 2.2.1 imply that the sum of two morphisms  $M \rightarrow N$  killing weights  $m, \dots, n$  kills these weights also; a direct proof of this fact is also very easy.
4. Similarly to part 3 of our theorem one can prove that  $\text{Mor}_{\cancel{[m, n]}} \underline{C}$  is a two-sided ideal of  $\text{Mor}(\underline{C})$  (cf. §1 of [Chr98]), i.e., that (in addition to the additivity properties  $\text{Mor}_{\cancel{[m, n]}} \underline{C}$  that were verified above) for any morphism  $j$  composable with  $g$  (either from the left or from the right) the corresponding composition kills weights  $m, \dots, n$  also.

In particular, if  $M \in \underline{C}_{w \notin [m, n]}$  then any  $\underline{C}$ -morphism from  $M$  kills weights  $m, \dots, n$ .

Besides, part 3 of the theorem can certainly be re-formulated as follows:  $\text{Mor}_{\cancel{[m', m-1]}} \circ \text{Mor}_{\cancel{[m, n]}} \underline{C} \subset \text{Mor}_{\cancel{[m', n]}} \underline{C}$  for any  $m' < m$ .

5. Certainly, part 4 of the theorem yields that weight-exact functors respect the condition of being without weights  $m, \dots, n$ , whereas weight-exact full embeddings "strictly respect" this condition. Hence weight-exact full embeddings of Karoubian categories also strictly respect the condition of an object to possess a decomposition avoiding weights  $m, \dots, n$ . This is also true for *weight-Karoubian* categories; see Proposition 3.1.3 below.

Now we relate  $\text{Mor}_{\cancel{[m, n]}} \underline{C}$  to weights for cohomology and to virtual  $t$ -truncations. We recall that for  $I \in \text{Obj } \underline{C}$  we denote the cohomological functor  $\underline{C}(-, I)$  (on  $\underline{C}$ ) by  $H_I$ .

**Theorem 2.2.3.** *Let  $g : M \rightarrow N$  be a  $\underline{C}$ -morphism,  $m \leq n \in \mathbb{Z}$ .*

*I. The following conditions are equivalent.*

1.  *$g$  kills weights  $m, \dots, n$ .*
2.  *$H(g)$  sends  $W^m(H)(N)$  inside  $W^{n+1}(H)(M)$  for any contravariant functor  $H : \underline{C} \rightarrow \underline{A}$ .*
3.  *$H(g)$  sends  $W^m(H_I)(N)$  inside  $W^{n+1}(H_I)(M)$  for all  $I \in \underline{C}_{w \geq m}$ .*
4.  *$H(g) = 0$  for  $H$  being any cohomological functor  $(\underline{C} \rightarrow \underline{A})$  of  $w$ -range  $[m, n]$ .*
5.  *$H(g) = 0$  for  $H = \tau^{\geq -n}(H_I)$  whenever  $I \in \underline{C}_{w \geq m}$ .*
6.  *$H(g) = 0$ , where  $H = \tau^{\geq -n}(H_{I_0})$  for  $I_0$  being some fixed choice of  $w_{\geq m}N$ .*

*II The following conditions are equivalent also.*

1.  *$M$  is without weights  $m, \dots, n$ .*
2.  *$H(M) = 0$  for  $H$  being any cohomological functor  $(\underline{C} \rightarrow \underline{A})$  of  $w$ -range  $[m, n]$ .*
3.  *$H(M) = \{0\}$  for  $H = \tau^{\geq -n}(H_I)$  whenever  $I \in \underline{C}_{w \geq m}$ .*
4.  *$H(M) = \{0\}$ , where  $H = \tau^{\geq -n}(H_{I_0})$  for  $I_0$  being some fixed choice of  $w_{\geq m}N$ .*

*Proof.* Certainly, condition I.2 implies condition I.3. Next, I.4 implies condition I.5 by Proposition 1.4.2(4), and the latter condition certainly implies condition I.6 (see Proposition 1.4.2(3)).

Now assume that  $g$  kills weights  $m, \dots, n$ . Then we have a commutative diagram

$$\begin{array}{ccc}
 M & \longrightarrow & w_{\geq n+1}M \\
 \downarrow g & & \downarrow j \\
 N & \xrightarrow{d} & w_{\geq m}N
 \end{array} \tag{2.2.2}$$

(it does not matter here whether we fix some choices of the rows or not). Applying  $H$  to this diagram, we obtain condition I.2.

Now fix some choice of the rows of (2.2.2) and take  $I$  being (any choice of)  $w_{\geq m}N$ . Assume that  $g$  fulfils condition I.3; then the morphism  $d \circ g$  belongs to the image of  $\underline{C}(w_{\geq n+1}M, w_{\geq m}N)$  in  $\underline{C}(M, w_{\geq m}N)$ . Thus  $g$  kills weights  $m, \dots, n$  (see Proposition 2.1.1(2)).

It remains to deduce condition I.4 from I.1, and deduce the latter one from condition I.6.

Assume that  $g$  kills weights  $m, \dots, n$ . If  $H$  is a cohomological functor of  $w$ -range  $[m, n]$  then the morphism  $H(w_{\geq m}N) \rightarrow H(N)$  is surjective and the morphism  $H(M) \rightarrow H(w_{\leq n}M)$  is injective (for any choices of the corresponding weight decompositions; see Proposition 1.4.2(5)). Since the composed morphism  $a : w_{\leq n}M \rightarrow w_{\geq m}N$  is zero (see Proposition 2.1.1(4); thus  $H(a) = 0$  also), we obtain condition I.4.

Now assume that condition I.6 is fulfilled. Consider the element  $r$  of the group  $\tau^{\geq -n}(H_{I_0})(N) = \text{Im}(\underline{C}(w_{\leq n+1}N, w_{\geq m}N) \rightarrow \underline{C}(w_{\leq n}N, w_{\geq m}N))$  obtained by composing the corresponding connecting morphisms. Since  $r$  vanishes in  $\tau^{\geq -n}(H_{I_0})(M) \subset \underline{C}(w_{\leq n}M, w_{\geq m}N)$ , the composed morphism  $a \in \underline{C}(w_{\leq n}M, w_{\geq m}N)$  is zero and we obtain condition I.1.

II Immediate from assertion I.

□

*Remark 2.2.4.* 1. Certainly, if there exists a  $t$ -structure  $t$  adjacent to  $w$  (see Definition 1.4.3(II)) then the cohomology functor  $H$  in parts I.6 and II.4 of our Theorem is isomorphic to  $\underline{C}(-, t^{\geq -n}(I_0))$  (see Remark 1.4.4(3)); note also that  $t^{\geq -n}(I_0) \in \underline{C}[m, n]$ .

More generally, here one may consider a certain  $\underline{C}' \supset \underline{C}$  endowed with a  $t$ -structure  $t'$  that is *orthogonal* to  $w$  (with respect to  $\underline{C}'(-, -)$ , i.e.,  $\underline{C}_{w \geq 0} \perp_{\underline{C}'} \underline{C}'^{\prime \prime \geq 1}$  and  $\underline{C}_{w \leq 0} \perp_{\underline{C}'} \underline{C}'^{\prime \prime \leq -1}$ ; see Definition 2.4.1(3) of [Bon13] or Definition 2.5.1(3) of [Bon10b]); then the obvious analogue of this statement is valid by Proposition 2.5.4(1) of *ibid*.

2. Now we prove that if  $t$  adjacent to  $w$  exists then the pair  $(\underline{C}[m, n], \text{Mor}_{[m, n]} \underline{C})$  yields an *injective class*, i.e., satisfies the duals to those conditions on the pair  $(\mathcal{P}, \mathcal{J})$  that were described in §2.3 of [Chr98]. So, we should check that the following statements are fulfilled.

(i) For a  $\underline{C}$ -morphism  $g$  we have  $g \in \text{Mor}_{[m, n]} \underline{C}$  if and only if  $H_I(g) = 0$  for all  $I \in \underline{C}[m, n]$ .

(ii) If  $M \in \text{Obj } \underline{C}$  and the functor  $H_M$  annihilates all elements of  $\text{Mor}_{[m, n]} \underline{C}$  then  $M \in \underline{C}[m, n]$ .

(iii) For any  $M \in \text{Obj } \underline{\mathcal{C}}$  there exist  $I_0 \in \underline{\mathcal{C}}[m, n]$  and  $h \in \underline{\mathcal{C}}(M, I_0)$  such that the corresponding morphism  $\text{Cone}(h)[-1] \rightarrow M$  kills weights  $m, \dots, n$ .

Now, (i) is immediate from part I of our theorem (see part 1 of this remark).

Next we verify (iii). We fix some  $w_{\geq m}M$ , denote  $t^{\geq -n}(w_{\geq m}M)$  by  $I_0$ , and complete the corresponding composed morphism  $h \in \underline{\mathcal{C}}(M, I_0)$  to a triangle

$$P \xrightarrow{g} M \xrightarrow{h} I_0. \quad (2.2.3)$$

Now,  $P$  is an extension of  $t^{\leq -n-1}(w_{\geq m}M)$  by  $w_{\leq m-1}M$  (by the octahedron axiom of triangulated categories). Since  $t^{\leq -n-1}(w_{\geq m}M) \in \underline{\mathcal{C}}_{w \geq n+1}$  (by the definition of adjacent structures),  $P$  is without weights  $m, \dots, n$  (by Theorem 2.2.1(7)). Thus  $g$  kills these weights (see Remark 2.2.2(4)).

Lastly, we deduce (ii) from the properties of (2.2.3). Consider  $M \in \text{Obj } \underline{\mathcal{C}}$  such that  $H_M$  annihilates all elements of  $\text{Mor}_{[m, n]} \underline{\mathcal{C}}$ . Then  $H_M(P) = 0$ ; hence  $M$  is a retract of  $I_0$  and we obtain the result.

Note also that the morphism classes  $\cap_{m \in \mathbb{Z}} \text{Mor}_{[m, m]} \underline{\mathcal{C}}$  are shift-stable and injective in this case (see Proposition 3.3 of [Chr98]); thus one may study the corresponding Adams spectral sequences (see §4 of *ibid.*).

3. So (in the setting of the previous part of this remark) for any  $M \in \text{Obj } \underline{\mathcal{C}}$  there exists a distinguished triangle  $P \rightarrow M \rightarrow I_0$  such that  $P \in \underline{\mathcal{C}}_{w \notin [m, n]}$  and  $I_0 \in \underline{\mathcal{C}}[m, n]$ . Hence  $\underline{\mathcal{C}}_{w \notin [m, n]}$  and  $\underline{\mathcal{C}}[m, n]$  yield a complete Hom-orthogonal pair in the sense of Definition 3.2 of [PoS16] (note however that we do not have to require the assumptions of Proposition 3.5 of *ibid.* to be fulfilled in our setting). Complete Hom-orthogonal pairs are interesting common generalizations of weight structures and  $t$ -structures; certain important properties of weight structures carry over to them. In particular, one can easily see that a  $\underline{\mathcal{C}}$ -morphism  $j$  whose target is  $M$  kills weights  $m, \dots, n$  if and only if it factors through  $g$ ; any morphism from  $M$  into an element of  $\underline{\mathcal{C}}[m, n]$  factors through  $h$  (cf. Proposition 1.2.3(7)). Possibly the author will study the relation of (certain) Hom-orthogonal pairs to injective classes of morphisms (including  $\text{Mor}_{[m, n]} \underline{\mathcal{C}}$ ) further in future. Note also that (in our setting) there exists a complete Hom-orthogonal pair *adjacent*

to  $(\underline{C}_{w \notin [m,n]}, \underline{C}[m,n])$ , i.e., that  $(\underline{C}[m,n], \underline{C}_{w \geq n+1} \oplus \underline{C}^{t \geq 1-m})$  is also a complete Hom-orthogonal pair (cf. Theorem 3.11 of [PoS16]; yet note once more that we do not have to assume its assumptions to be fulfilled in our case).

Besides we note that the axiomatics of complete Hom-orthogonal pair easily yields the following:  $\underline{C}_{w \notin [m,n]}$  can be completed to the data of a projective class of morphisms; one may say that this class is "complementary" to  $(\underline{C}[m,n], \text{Mor}_{[m,n]} \underline{C})$ . This projective class can also be described at the *product* of the projective class corresponding to  $\underline{C}^{t \leq -n-1} = \underline{C}_{w \geq n+1}$  with the one corresponding to  $\underline{C}_{w \leq m-1}$  (see Proposition 3.3 of [Chr98]).

Another related notion is the natural  $t$ -structure analogue of killing weights; so, for  $t$  being a  $t$ -structure for  $\underline{C}$  and  $g \in \underline{C}(M, N)$  one may ask whether the corresponding composed morphism  $t^{\leq n} M \rightarrow N$  factors through  $t^{\leq m-1} M$  (for  $m \leq n \in \mathbb{Z}$ ). Note that this setting is closely related to ghost morphisms as studied in §7 of [Chr98].

4. We are (mostly) interested in "complicated" triangulated categories (yet see Remark 1.4.4(5)). So for  $g \in \underline{C}(M, N)$  it can be quite difficult to check whether  $g$  kills weights  $m, \dots, n$  (or if  $M \in \underline{C}_{w \notin [m,n]}$ ) using (any of the versions of) the definition of these notions. Yet Theorem 2.2.3 yields a way of checking that  $g$  **does not** kill weights  $m, \dots, n$  (resp.  $M \notin \underline{C}_{w \notin [m,n]}$ ) by looking at a single cohomology theory on  $\underline{C}$ ; certainly, one can choose an "easily computable" one here.

A more complicated problem is to find a "reasonable" detecting family of cohomology theories (see Remark 2.1.3(2)). Certainly,  $\underline{C}[m,n]$  yields a collection of this sort if a  $t$ -structure adjacent to  $w$  exists (see part 1 of this remark); we will use this observation in §2.4 below. Yet one can be interested in finding a smaller collection of ("nice") detector theories that does not depend on the choice of  $N$  (whereas  $\tau^{\geq -n}(H_{I_0})$  can be somewhat difficult to compute).

5. In several previous papers of the author (see also [Heb11]) certain *Chow* weight structures for various motivic categories were constructed; their hearts were certain categories of Chow motives. Note that in all of these examples there exists a "big" motivic category  $DM$  that is closed with respect to all coproducts; inside it there is its subcategory  $DM^c$  of

compact objects (whose objects are usually called constructible or geometric motives). Next, inside  $DM^c$  there is an (additive Karoubian) negative category  $\underline{B}$  of the corresponding Chow motives that strongly generates it. Thus we have a "geometric" Chow weight structure  $w_{Chow}^c$  for  $DM^c$  (whose heart is just  $\underline{B}$ ) and the "big" Chow structure  $w_{Chow}$  for  $DM$  (see [Bon14], [BoI15], or Remark 1.4.4(4); certainly the embedding  $DM^c \rightarrow DM$  is weight-exact). Whereas one is usually interested in  $w_{Chow}^c$  only, it seems that the corresponding adjacent structure  $t_{Chow}$  (see the aforementioned remark once more) does not (usually) restrict to  $DM^c$  (note also that  $\underline{Ht}_{Chow}$  is isomorphic to the category of additive contravariant functors  $\underline{B} \rightarrow Ab$ ). Now, to detect whether a  $DM^c$ -morphism  $g$  kills weights  $m, \dots, n$  it suffices to compute  $H_I(g)$  for  $I$  running through  $t^{\geq -n}(DM_{w_{Chow}^c \geq m}^c) \subset DM[m, n]$ .

6. Still we would certainly prefer to use some "classical" cohomology of motives instead. We try to describe the corresponding picture here.

First we recall that for motives over a field the (Chow-)weight filtrations for étale and singular cohomology (with rational coefficients) for (motives of) varieties differ from Deligne's ones only by a shift; see Remark 2.4.3 of [Bon10a] and Proposition 2.5.1 of [Bon15]; a natural analogue of this statement for motives over varieties was established in [Bon14]. Moreover, the weights of étale (co)homology conjecturally "detect weights" of  $\mathbb{Q}$ -linear motives over an arbitrary base; see Proposition 3.3.1(4) of [Bon15]. Yet this does not imply that étale (or singular) cohomology detects whether a morphism of motives kills certain weights; in particular, note that a non-zero morphism of Chow motives can certainly yield zero on cohomology. One can only hope that composing a "long" chain of motivic morphisms that kill certain weights in étale cohomology necessarily yields a morphism that kills weights in a certain range (in the sense of our definition). The situation is somewhat better for mixed Tate motives (see Remark 2.4.4(2) below).

7. On the other hand, singular and étale cohomology of motives conjecturally detects whether a given motif is without weights  $m, \dots, n$  (cf. Theorem 1.11 of [Wil15b]); Wildeshaus has also established several non-conjectural cases of this statement unconditionally (see Theorem 3.4 of [Wil15b] and Theorem 1.13(d) of [Wil15a]).

8. Applying categorical duality one certainly obtains that  $g \in \text{Mor}_{[m,n]} \underline{C}$  if and only if  $H(g) = 0$ , where  $H : \underline{C} \rightarrow \text{Ab}$  is the homological functor  $O \mapsto \text{Im}(\underline{C}(w_{\leq n}M, w_{\geq m-1}O) \rightarrow \underline{C}(w_{\leq n}M, w_{\geq m}O))$ .
9. Certainly, part I of our theorem implies parts 1 and 2 of Theorem 2.2.1 (cf. also Remark 2.2.2(4)); so we obtain an alternative proof of these statements.

### 2.3 Relation to the weight complex functor (and its conservativity)

Now we relate the properties studied in the previous subsection with the weight complex functor.

**Proposition 2.3.1.** *Let  $g \in \underline{C}(M, N)$  (for some  $M, N \in \text{Obj } \underline{C}$ );  $m \leq n \in \mathbb{Z}$ . Then the following statements are valid.*

1.  *$g$  kills weight  $m$  if and only if  $t(g) \smile_{[-m, -m]} 0$  (in the notation of Remark 1.3.3(3); recall that this property does not depend on the choice of  $t(f)$ ).*
2. *If  $f_i$  for  $n \geq i \geq m$  is a chain of composable  $\underline{C}$ -morphisms such that  $t(f_i) \smile_{[-i, -i]} 0$  for all  $i$ , then  $f_n \circ f_{n-1} \circ \cdots \circ f_m$  kills weights  $m, \dots, n$ .*
3.  *$M$  is without weights  $m, \dots, n$  if and only if  $t(id_M) \smile_{[-n, -m]} 0$ .*

*Proof.* 1. If  $g$  kills weight  $m$  then we can choose  $t(g)$  such that the component  $t(g)^{-m}$  is zero (see Proposition 2.1.1(6)). Conversely, if  $t(g) \smile_{[-m, -m]} 0$  then we can assume  $t(g)^{-m} = 0$  (see Remark 1.3.3(4)); hence  $g$  kills weight  $m$  (see Proposition 2.1.1(7)).

2. Immediate from the previous assertion combined with Theorem 2.2.1(3).

3. If  $M \in \underline{C}_{w \notin [m, n]}$  then we can choose  $t(id_M)$  so that  $t(id_M)^i = 0$  for all  $i$  between  $-n$  and  $-m$  (this is an easy consequence of Proposition 2.1.1(6)); hence  $t(id_M) \smile_{[-n, -m]} 0$ . Conversely, if  $t(id_M) \smile_{[-n, -m]} 0$ , then  $t(id_M) \smile_{[i, i]} 0$  for all  $i$  between  $-m$  and  $-n$ ; thus applying the previous assertion to the composition  $id_M^{n-m+1}$  we obtain that  $M$  is without weights  $m, \dots, n$ .

□



*Remark 2.3.2.* 1. If  $M$  possesses a decomposition avoiding weights  $m, \dots, n$  then Proposition 1.3.2(3) yields that a similar decomposition exists for  $t(M)$  (i.e., we can assume that the weight complexes of  $X$  and  $Y$  in (2.2.1) are concentrated in degrees  $\geq 1 - m$  and  $\leq -1 - n$ , respectively). Next, if  $t(M)$  possesses a decomposition satisfying this condition then  $M \in \underline{C}_{w \notin [m, n]}$  by part 3 of our proposition. Hence these three conditions are equivalent if  $\underline{C}$  is Karoubian; see Theorem 2.2.1(8).

2. Certainly, part 2 of our proposition is a vast generalization of (the nilpotence statement in) Theorem 3.3.1(II).

3. A rich collection of triangulated categories endowed with weight structures can be described using *twisted complexes* (in the sense defined in [BoK90]) over a *negative differential graded category*  $C$  (see §6.1–2 of [Bon10a]). We recall that for  $\underline{C} = Tr(C)$  a  $\underline{C}$ -morphism  $h$  between two twisted complexes  $(P^i, q_{ij}), (P'^i, q'_{ij}) \in \text{Obj } \underline{C}$  is given by a certain collection of arrows  $h_{ij} \in C^{i-j}(P^i, P'^j)$  (satisfying certain "closedness" conditions and considered up to "homotopies" of a certain sort); we have  $h_{ij} = 0$  if  $i > j$  since  $C$  is negative. Then one can easily check that  $h$  kills weights  $m, \dots, n$  if and only if it is homotopic to a morphism  $h'$  such that  $h'_{ij} = 0$  (also) if  $-n \leq i \leq j \leq m$ .

Next, the composition of morphisms in  $Tr(C)$  is given by the "obvious" (i.e., the "matrix-like") compositions of the corresponding collections of arrows (so, it does not take the differentials in  $C$  into account). Thus Theorem 2.2.1(3) (cf. also Remark 2.2.2(4)) in this case reduces to the corresponding trivial property of (lower triangular) matrices.

Lastly, note that  $t(h)$  (in this case) can be described by the collection  $h_{ii}$  (for  $i \in \mathbb{Z}$ ); this gives an illustration for part 1 of Proposition 2.3.1.

4. Combining part 1 of our proposition with Theorem 2.2.3(I.6) and applying them to the stupid weight structure for  $K(\underline{B})$  one can re-prove Theorem 2.1 of [Bar05]; the corresponding detector functor coincides with the one described in loc. cit. (cf. Remark 1.4.4(6)).

Now we are able to improve the ("partial") conservativity property of weight complexes given by Theorem 3.3.1(V) of [Bon10a]. We need some definitions; our choice of conventions is motivated by the fact that we consider cohomological complexes only (in this paper).

**Definition 2.3.3.** 1. We will call the elements of  $\cap_{i \in \mathbb{Z}} \underline{C}_{w \leq i}$  (resp. of  $\cap_{i \in \mathbb{Z}} \underline{C}_{w \geq i}$ ) *right degenerate* (resp. *left degenerate*).

2. We will say that  $w$  is *non-degenerate* if  $\cap_{i \in \mathbb{Z}} \underline{C}_{w \leq i} = \cap_{i \in \mathbb{Z}} \underline{C}_{w \geq i} = \{0\}$ .

3. We will say that  $M \in \text{Obj } \underline{\mathcal{C}}$  is *w-degenerate* (or weight-degenerate) if  $t(M) \smile 0$ .

Now we relate *w-degenerate* objects to right and left degenerate ones.

**Theorem 2.3.4.** *Let  $g : M \rightarrow M'$  be a  $\underline{\mathcal{C}}$ -morphism. Then the following statements are valid.*

I.1.  *$t(g)$  is an isomorphism if and only if  $\text{Cone}(g)$  is a *w-degenerate* object.*

2. *Any extension of a left degenerate object of  $\underline{\mathcal{C}}$  by a right degenerate one is *w-degenerate*.*

3. *If  $M$  is an extension of a left degenerate object by an element of  $\underline{\mathcal{C}}_{w \leq 0}$  (resp. is an extension of an element of  $\underline{\mathcal{C}}_{w \geq 0}$  by a right degenerate object) then  $t(M) \smile^w 0$  (resp.  $t(M) \smile_w 0$ ; see Remark 1.3.3(5)).*

II If  $\underline{\mathcal{C}}$  is Karoubian then the statements converse to assertions I.2 and I.3 are also valid. Being more precise,  $M$  is *w-degenerate* if and only if it is an extension of a left degenerate object by a right degenerate one;  $t(M) \smile^w 0$  (resp.  $t(M) \smile_w 0$ ) if and only if  $M$  is an extension of a left degenerate object by an element of  $\underline{\mathcal{C}}_{w \leq 0}$  (resp. is an extension of an element of  $\underline{\mathcal{C}}_{w \geq 0}$  by a right degenerate object).

III If  $\underline{\mathcal{C}}$  contains no non-trivial left degenerate (resp. right degenerate) objects then its class of weight-degenerate objects coincides with the one of right degenerate (resp. left degenerate) ones. In particular, if  $w$  is non-degenerate then  $t(M) \neq 0$  for non-zero objects.

*Proof.* I.1. Immediate from Proposition 1.3.2(3,5).

2. If  $N$  is left (resp. right) degenerate then we can take  $w_{\leq l} N = 0$  (resp.  $w_{\leq l} N = N$ ) for all  $l \in \mathbb{Z}$ . Hence one of the choices of  $t(N)$  is 0; thus the same is true for extensions in question according to the previous assertion.

3. Immediate from assertion I.1 combined with 1.3.2(3).

II. We investigate when  $t(M) \smile^w 0$ .

For any  $n > 0$  we have  $\text{id}_{t(M)} \smile_{[-n, -1]} 0$  (see Remark 1.3.3(5)).

Since  $\underline{\mathcal{C}}$  is Karoubian, for any  $n > 0$  there exists a distinguished triangle  $X_n \rightarrow M \rightarrow Y_n$  with  $X_n \in \underline{\mathcal{C}}_{w \leq 0}$ ,  $Y_n \in \underline{\mathcal{C}}_{w \geq n+1}$  (see Remark 2.3.2). All of these triangles are isomorphic to the one for  $n = 1$  by the uniqueness statement in Theorem 2.2.1 (7). Hence  $Y_1$  is left degenerate and we obtain a triangle of the sort desired.

III Certainly, the "in particular" part of the assertions follows from the (combination of the two) remaining statements. Both of the latter follow

immediately from assertion II if  $\underline{C}$  is Karoubian. Lastly, in the general case if  $M$  is degenerate and  $\underline{C}$  contains no non-trivial left degenerate (resp. right degenerate) objects then  $M$  is a retract of a right degenerate (resp. left degenerate) object by Corollary 3.1.5 below. Hence  $M$  is right degenerate (resp. left degenerate) itself.  $\square$

*Remark 2.3.5.* 1. So we get a precise answer to the question when  $t(g)$  is an isomorphism in the Karoubian case (in the general case one should combine part I.1 of the theorem with Corollary 3.1.5 below). In particular, the weight complex functor is conservative if  $w$  is non-degenerate; this is a significant improvement of Theorem 3.3.1(V) of [Bon10a] (that states that the restrictions of  $t$  to the subcategories of left and right bounded objects of  $\underline{C}$  are conservative under this condition).

2. In Corollary 3.1.5 several equivalent conditions for  $t(M) \smile^w 0$ ,  $t(M) \smile_w 0$ , and  $t(M) = 0$  (for  $\underline{C}$  being not necessarily Karoubian) will be formulated.

3. One can easily check that the motif constructed in Lemma 2.4 of [Ayo15] is  $w_{Chow}$ -degenerate; here one can use the fact that  $t$  commutes with countable homotopy colimits in this case (by the dual to Theorem 2.2.6(II.6) of [Bon13]). On the other hand, none of the versions of  $DM$  contains non-zero left degenerate objects (for the corresponding  $w_{Chow}$ ; see Remark 2.2.4(5)) since  $DM$  is weakly generated by the corresponding Chow motives (in the sense described in §1.1). Thus there exist non-zero  $w_{Chow}$ -right degenerate objects in  $DM$  (at least) when the base scheme is a big enough field and the coefficient ring is non-torsion (one can certainly generalize this to motives over other base schemes); a minor modification of the argument from (Lemma 2.4 of) [Ayo15] yields the corresponding example for torsion coefficients also.

## 2.4 The case of the spherical weight structure on the stable homotopy category

Now we apply our results to the stable homotopy category  $SH$  (whose detailed description can be found in [Mar83]; certainly, it is a Karoubian triangulated category) and the *spherical* weight structure for it defined in §4.6 of [Bon10a]. We denote the sphere spectrum (in  $SH$ ) by  $S^0$ . We recall that the category  $\underline{B}$  of finite coproducts of  $S^0$  satisfies the conditions formulated in Remark 1.4.4(4). The corresponding  $t$ -structure is the Postnikov  $t$ -structure

$t_{Post}$  whose heart is isomorphic to  $Ab$ ; the corresponding  $t_{Post}$ -homology functor is given by  $\pi_0 = SH(S^0, -)$ . Besides,  $SH^{t_{Post} \geq -n}$  (for any  $n \in \mathbb{Z}$ ) is the class of  $n - 1$ -connective spectra. Certainly,  $SH[0, 0]$  contains exactly the Eilenberg-MacLane spectra  $HG$ , where  $G$  runs through all abelian groups. We denote  $SH(M, HG)$  by  $H^i(M, HG)$ . Besides, our notation  $SH[m, n]$  (see Definition 1.4.3(II)) in this setting is just the one introduced in §3.2 of [Mar83].

In order to make the notation in the current paper compatible with the one used earlier, we will use the following (somewhat weird) notation for  $H\mathbb{Z}$ -homology: (for  $M \in \text{Obj } SH$ ) we will denote  $SH(S^0, M \wedge H\mathbb{Z}[i])$  by  $H_i^{EM, \mathbb{Z}}(M)$  (so, it is concentrated in negative degrees if  $M$  is a connective spectrum).

Now we recall the main results of §4.6 of [Bon10a].

**Proposition 2.4.1.** *Let  $M \in \text{Obj } SH$ ,  $i \in \mathbb{Z}$ . Then the following statements are valid.*

1.  *$SH$  is endowed with a certain spherical weight structure  $w^{sph}$  such that  $SH_{w^{sph} \geq 0} = SH^{t_{Post} \leq 0} = (\cup_{i < 0} S^0[i])^\perp$ .*
2.  *$SH_{w^{sph} = 0}$  consists of all small coproducts of (copies of)  $S^0$ , whereas  $\underline{H}w^{sph}$  is equivalent to  $FAb$  (the category of free abelian groups). The comparison functor is given by  $SH(S^0, -)$ ; it sends  $S^0$  into  $\mathbb{Z}$ .*
3. *The weight complex functor  $t$  is actually an exact functor  $SH \rightarrow K(FAb) \cong D(Ab)$ .*
4.  *$H_i(t(M)) \cong H_i^{EM, \mathbb{Z}}(M)$ .*
5.  *$H^i(M, HG)$  is naturally isomorphic to the  $i$ th homology of the complex  $H(Ab(X^{-*}, G))$  for any abelian group  $G$ .*
6.  *$w^{sph}$  can be restricted to the (triangulated) subcategory  $SH_{fin}$  of finite spectra.*

*Proof.* These statements were proved in §4.6 of [Bon10a] (yet see the remark below). □

- Remark 2.4.2.* 1. The proof of our assertion 4 in (the published version of) [Bon10a] contains a substantial gap: the spectral sequence argument used there only works if  $M$  is bounded below (by Theorem 2.3.2(II(iii)) of *ibid.*). Note still that any  $M \in \text{Obj } SH$  can be presented as the *countable homotopy colimit* of its  $t_{Post}$ -truncations (from above; cf. the proof of Theorem 4.5.2(I.2) of *ibid.*). Since both the left and the right hand side of the assertion yield homological functors from  $SH$  into  $Ab$  that commute with all small coproducts (the latter property is easy and well-known for  $H\mathbb{Z}$ -homology, and can be easily verified for  $H_i(t(M))$ ; one can apply the dual to Theorem 2.2.6(II.6) of [Bon13] here), they also respect (countable) homotopy colimits and we obtain the result.
2. Note that any object  $N$  in  $K_w(FAb) \cong D(Ab)$  functorially splits as the coproduct of  $H_i(N)[-i]$  (for  $i \in \mathbb{Z}$ ). Hence we can also present  $N$  (in  $K(FAb)$ ) as

$$\coprod_{i \in \mathbb{Z}} (A^{i-1} \xrightarrow{f^i} B^i)[-i], \quad (2.4.1)$$

where  $f^i$  are certain embeddings of free abelian groups (and we put  $A^j$  and  $B^j$  in degree  $j$  for  $j \in \mathbb{Z}$ ). Furthermore, the homology of  $t(M)$  is functorially isomorphic to  $H_*^{EM, \mathbb{Z}}(M)$  (for  $M \in \text{Obj } SH$ , by Proposition 2.4.1(4)). Besides, for any  $g \in SH(M, N)$  we have a well-defined class of  $t(g)$  in  $Ab(H_i^{EM, \mathbb{Z}}(M), H_{i-1}^{EM, \mathbb{Z}}(N))$  (for  $M, N \in \text{Obj } SH$  and any  $i \in \mathbb{Z}$ ).

3. For  $X \in \text{Obj } SH$  and all  $n \in \mathbb{Z}$  we take  $X^{(n)}$  being (any choice of)  $w_{\leq n}^{sph} X$ . We connect  $X^{(n)} \rightarrow X^{(n+1)}$  by the unique morphisms  $i^n$  "compatible with  $id_X$ " (see Remark 1.2.4(1)); then  $\text{Cone}(i^n)$  is a coproduct of  $S^0[n+1]$  (see Proposition 1.2.3(8)), i.e., of  $n+1$ -dimensional spheres in  $SH$ . Next, one can easily check that  $X$  is the *minimal weak colimit* of  $X^{(n)}$  (see Proposition 3.3 of [Mar83]). Furthermore, part 4 of our proposition implies that the inverse limit of the  $H\mathbb{Z}$ -homology of  $X^{(n)}$  (with respect to  $i^n$ ) vanishes. Hence  $X^{(n)}$  give a *cellular tower* for  $X$  (in the sense of the beginning of §6.3 of [Mar83]). Conversely, if  $X^{(n)}$  is (a term of) a certain cellular tower for  $X$  (i.e., an  $n$ -skeleton of  $X$  in the terms of *loc. cit.*) then  $X^{(n)} \in SH_{w_{\leq n}^{sph}}$  (by Proposition 2.4.3(6) below) and  $\text{Cone}(X^{(n)} \rightarrow X) \in SH_{w_{\geq n+1}^{sph}}$  (since this cone is the minimal weak colimit of  $\text{Cone}(X^{(n)} \rightarrow X^{(l)}) \in SH_{w_{\geq n+1}^{sph}}$  for  $l \geq n$ ). Hence  $SH_{w_{\leq n}^{sph}}$

consists exactly of  $n$ -skeleta (of certain spectra; cf. also Definition 6.7 of [Chr98]) and all possible cellular towers of  $X$  come from some choices of  $w_{\leq n}^{sph} X$ . This statement was made in §4.6 of [Bon10a]; thus we have justified it completely in the current paper. As a consequence we obtain that the  $w^{sph}$ -weight spectral sequences (for (co)homological functors defined on  $SH$ ; see §2.3–2.4 of *ibid.* or §2.3 of [Bon13]) are actually Atiyah-Hirzebruch ones (and we can compute them using arbitrary cellular towers; we certainly also have a similar statement for  $w^{sph}$ -filtrations).

4. Since the category  $\underline{B}$  (as well as its strong generator  $S^0$ ) weakly generates  $SH$ , part 1 of the proposition yields that  $SH$  contains no non-trivial left degenerate objects (i.e., that  $\cap_{i \in \mathbb{Z}} SH_{w^{sph} \geq i} = \{0\}$ ). Now we will use this observation to obtain a (less trivial) description of all degenerate objects in  $SH$ .

We use the results of the current paper to prove some new properties  $SH$  (none of them were formulated in [Bon10a]; yet assertion 6 of our proposition is equivalent to Proposition 6.8 of [Chr98] by Remark 2.4.2(3)).

**Proposition 2.4.3.** *Let  $M, N \in \text{Obj } SH$ ,  $g \in SH(M, N)$ ,  $m \leq n \in \mathbb{Z}$ . Then the following statements are valid.*

1.  $g$  kills  $w^{sph}$ -weight  $n$  if and only if  $H_{-n}^{EM, \mathbb{Z}}(g) = 0$  and the class of  $g$  in  $\text{Ab}(H_{-n}^{EM, \mathbb{Z}}(M), H_{-n-1}^{EM, \mathbb{Z}}(N))$  (see Remark 2.4.2(2)) vanishes.
2.  $g \in \text{Mor}_{[m, n]} SH$  if and only if  $H(g) = 0$  for any  $H$  representable by an element of  $SH[m, n]$ . Moreover, if  $g$  is an  $SH_{fin}$ -morphism then it suffices to consider elements of  $SH[m, n]$  with finitely generated  $H\mathbb{Z}$ -homology here only.
3.  $M$  is without weights  $m, \dots, n$  if and only if  $H_i^{EM, \mathbb{Z}}(M) = 0$  for  $-n \leq i \leq -m$  and  $H_{-m+1}^{EM, \mathbb{Z}}(M)$  is a free abelian group.
4. For any  $M \in \text{Obj } SH$  there exists a distinguished triangle  $P \xrightarrow{g} M \xrightarrow{h} I_0$  such that  $I_0 \in SH[m, n]$  and  $P$  is without weights  $m, \dots, n$ . Moreover, an  $SH$ -morphism  $j$  whose target is  $M$  kills weights  $m, \dots, n$  if and only if it factors through  $g$  (from this triangle); any morphism from  $M$  into an element of  $SH[m, n]$  factors through (this)  $h$ .

5. *The class of weight-degenerate objects of  $SH$  is the one of acyclic spectra (i.e., of those with vanishing  $H\mathbb{Z}$ -homology); it coincides with the class of right degenerate spectra.*
6.  *$M \in SH_{w^{sph} \leq n}$  if and only if  $H_i^{EM, \mathbb{Z}}(M) = 0$  for all  $i \leq -n$  and  $H_{-n+1}^{EM, \mathbb{Z}}(M)$  is a free abelian group.*
7.  *$H_i^{EM, \mathbb{Z}}(M) = 0$  for all  $i > -n$  if and only if  $M$  is an extension of an object of  $SH_{w^{sph} \geq n} = SH^{t_{Post} \geq -n}$  by an acyclic spectrum. Moreover, these two conditions are equivalent to the vanishing of  $H^i(M, H(\mathbb{Q}/\mathbb{Z}))$  for all  $i < n$ .*

*Proof.* 1. By Proposition 2.3.1(1), we should check whether  $t(g) \smile_{[-n, -n]} 0$ . Thus the assertion is an easy consequence of Remark 2.4.2(2).

2. The first part of the assertion is immediate from Theorem 2.2.3(I) (see also Remark 2.2.4(1)). We obtain the second part by noting that any finite spectrum possesses an  $(m-)w^{sph}$ -decomposition whose components are finite, whereas the  $H^{EM, \mathbb{Z}}$ -homology groups of finite spectra are finitely generated.

3. According to Proposition 2.3.1(3), we should check whether  $t(id_M) \smile_{[-n, -m]} 0$ . For this purpose it suffices to apply Remark 2.4.2(2) again.

4. See Remark 2.2.4(2–3).

5. Since  $SH$  is Karoubian and contains no non-trivial left degenerate objects (see Remark 2.4.2(4)), all of its weight-degenerate objects are right degenerate (by Theorem 2.3.4(III)). Next, since  $K_w(\underline{H}w^{sph}) \cong D(Ab)$ , weight-degenerate spectra are exactly the acyclic ones (see Proposition 2.4.1(4)).

6. The proof is similar to that of assertion 3; one should apply Theorem 2.3.4(II) (instead of Proposition 2.3.1(3)) and recall again that  $SH$  contains no non-trivial left degenerate objects.

7. Once more, to prove the first part of the assertion we should combine Theorem 2.3.4(II) with Remark 2.4.2(2). To prove the second part it suffices to note that  $\mathbb{Q}/\mathbb{Z}$  is an injective cogenerator of  $Ab$  and apply Proposition 2.4.1(5).

□

- Remark 2.4.4.* 1. Non-zero acyclic objects do exist; see Theorem 16.17 of [Mar83] (cf. also the discussion at <http://mathoverflow.net/questions/218148/does-the>
2. So, we obtain a "reasonable cohomological description" of  $\text{Mor}_{[m,n]} SH$ . Besides, if certain composable morphisms  $g_n$  satisfy the (equivalent) conditions of part 1 of our proposition for all  $l \geq n \geq m$  (where  $l \geq m \in \mathbb{Z}$ ) then the morphism  $h = g_l \circ g_{l-1} \circ \cdots \circ g_m$  kills  $w^{sph}$ -weights  $m, \dots, l$ . Next one can consider the corresponding version of (2.1.3) for any choices of cellular filtrations for the source and the target of  $h$  (see Remark 2.4.2(3)).
  3. As it often happens with (author's) results related to weight structures, we did not use the ("full") definition of  $SH$  for proving its properties described above. It suffices to have a category  $\underline{B}$  as in Remark 1.4.4(4) such the corresponding category  $\underline{Ht} \cong \text{AddFun}(\underline{B}^{op}, Ab)$  is of projective dimension 1 (in particular, this is the case if  $\underline{B}$  consists of finite coproducts of a single object  $S$  and the ring  $\underline{C}(S, S)$  is hereditary). Then we have  $K_w(\underline{Hw}) \cong D(\underline{Ht})$  (see Remark 3.3.4 of [Bon10a]), and one can easily prove the natural analogues of all the results of this section (though the corresponding "homology" and "cohomology" of weight complexes does not have any "topological" significance in general).
  4. In particular, inside the category  $DM$  of motives over any perfect field (see §4.2 of [Deg11]) one can take for  $\underline{C}$  its *localizing subcategory*  $DTM$  *generated by* the Tate motives  $\mathbb{Z}(i)$  for  $i \in \mathbb{Z}$  (i.e., by its smallest triangulated subcategory containing all  $\mathbb{Z}(i)$  and closed with respect to small coproducts). Note also that this category possesses two important weight structures: the heart of the first ("Chow") one is "weakly generated" by  $\mathbb{Z}(i)[2i]$  (and it is "compatible" with Chow weight structure for the whole  $DM$ ), whereas the second ("Gersten") heart is weakly generated by  $\mathbb{Z}(i)[i]$  (cf. §4.5 of [Bon13]).

One can also consider Artin-Tate motives here (cf. [Wil08]); instead of motives with integral coefficients one can take  $R$ -linear ones for  $R$  being any Dedekind domain (cf. [MVW06]).



### 3 On generalizations to non-Karoubian categories and applications

This section is mainly dedicated to the extension of our main results to non-Karoubian triangulated categories and to their applications.

In §3.1 we discuss certain extensions of the results of §2 to the case where  $\underline{C}$  is not (necessarily) Karoubian (they are mostly "generalizations up to retracts"). Our main tool is the existence of a triangulated category  $\underline{C}'$  containing  $\underline{C}$  such that  $\underline{C}$  is dense in  $\underline{C}'$ ,  $w$  extends onto it and the heart of this extension  $w'$  is Karoubian (we call such a couple  $(\underline{C}', w')$  a *weight-Karoubian extension* of  $\underline{C}$ ).

In §3.2 we prove that we can check whether  $M \in \underline{C}_{w \geq -m}$  (or  $M \in \underline{C}_{w \leq -m}$ ) using certain "pure" (co)homology theories; these results are applied in [BoT15].

In §3.3 we construct certain counterexamples to demonstrate that the modifications made in §3.1 to "adjust" the results of §2 to the non-Karoubian case cannot be avoided.

#### 3.1 On weight-Karoubian extensions and generalizations of our results to non-Karoubian categories

We recall the central definitions of [BoS16].

**Definition 3.1.1.** 1. We will call a triangulated category  $\underline{C}'$  an *idempotent extension* of  $\underline{C}$  if it contains  $\underline{C}$  and there exists a fully faithful exact functor  $\underline{C}' \rightarrow \text{Kar}(\underline{C})^2$ .

2. We will say that a weight structure  $w$  extends onto an idempotent extension  $\underline{C}'$  of  $\underline{C}$  whenever there exists a weight structure  $w'$  for  $\underline{C}'$  such that the embedding  $\underline{C} \rightarrow \underline{C}'$  is weight-exact. In this case we will call  $w'$  an *extension* of  $w$ .

3. We will say that a triangulated category  $\underline{C}'$  endowed with a weight structure  $w'$  is *weight-Karoubian* if  $Hw'$  is Karoubian.

---

<sup>2</sup>Recall that (according to Theorem 1.5 of [BaS01]) the category  $\text{Kar}(\underline{C})$  can be naturally endowed with the structure of a triangulated category so that the natural embedding functor  $\underline{C} \rightarrow \text{Kar}(\underline{C})$  is exact. Hence  $\underline{C}'$  is an idempotent extension of  $\underline{C}$  if and only if any object of  $\underline{C}'$  is a retract of some object of  $\underline{C}$  and  $\underline{C}$  is dense (see §1.1) in  $\underline{C}'$ .

4. We will call a weight-Karoubian category  $(\underline{C}', w')$  a *weight-Karoubian extension* of  $(\underline{C}, w)$  if  $\underline{C}'$  is an idempotent extension of  $\underline{C}$  and  $w'$  is the extension of  $w$  onto it.

Now we recall those results of *ibid.* that are relevant for the current paper.

**Proposition 3.1.2.** 1. Let  $\underline{C}'$  be an idempotent extension of  $\underline{C}$  such that  $w$  for extends to a weight structure  $w'$  on it. Then  $\underline{C}'_{w \geq 0}$  (resp.  $\underline{C}'_{w' \leq 0}$ , resp.  $\underline{C}'_{w'=0}$ ) is the Karoubi-closure of  $\underline{C}_{w \geq 0}$  (resp.  $\underline{C}_{w \leq 0}$ , resp.  $\underline{C}_{w=0}$ ) in  $\underline{C}'$ .  
2. Any  $(\underline{C}, w)$  possesses a weight-Karoubian extension.

*Proof.* 1. This is Theorem 2.2.2(I.1) of *ibid.*

2. The statement is given part III.1 of *loc. cit.*

□

The following observations is crucial for this section.

**Proposition 3.1.3.** The conclusions of Proposition 1.2.3(6), Theorem 2.2.1(8), and Theorem 2.3.4(II) remain valid if we assume  $\underline{C}$  to be weight-Karoubian (only).

*Proof.* It suffices to verify that the first of the statements mentioned can be generalized this way, since then the proofs of the other two facts (given above) would extend to the weight-Karoubian case automatically.

The idea is to construct the retracts mentioned in Proposition 1.2.3(6) inside  $\text{Kar}(\underline{C})$ , and prove then that they are actually isomorphic to objects of  $\underline{C}$ . So, we consider the  $\text{Kar}(\underline{C})$ -decomposition  $w_{\leq m}M \cong M_1 \oplus M_0$  corresponding to  $h$ . Since  $\underline{Hw}$  is Karoubian, it suffices to verify that  $M_0$  is a retract of  $M_m = w_{\geq m}(w_{\leq m}M) \in \underline{C}_{w=m}$  (see part 7 of the proposition). Since  $M_0$  is a retract of  $M_{w \geq m+1}[-1] \in \underline{C}_{w \geq m}$ , we have  $\underline{C}_{w \leq m-1} \perp M_0$ . Thus if we apply the functor  $\text{Kar}(\underline{C})(-, M_0)$  to the distinguished triangle  $\dots w_{\leq m-1}M (= w_{\leq m-1}(w_{\leq m}M)) \rightarrow w_{\leq m}M \rightarrow M_0$  then the long exact sequence obtained yields that the projection  $w_{\leq m}M \rightarrow M_0$  factors through  $M_m$ . Certainly,  $\text{id}_{M_0}$  possesses this property also.

□

*Remark 3.1.4.* In [BoS16] much more information on idempotent extensions of  $\underline{C}$  such that  $w$  extends to them is contained. In particular, the (essentially) minimal weight-Karoubian extension of  $\underline{C}$  was described as follows:  $\text{Kar}_{\min}^w(\underline{C}) = \langle \text{Obj Kar}(\underline{C}^-) \cup \text{Obj Kar}(\underline{C}^+) \rangle_{\text{Kar}(\underline{C})}$ . Since it is minimal, applying our proposition to it gives the maximal possible amount of information on the corresponding  $\underline{C}$ .

Now we use Proposition 3.1.3 for deducing a certain version of Theorem 2.3.4(II) that would be valid for a not (necessarily) Karoubian  $\underline{\mathcal{C}}$ .

**Corollary 3.1.5.** *Let  $M \in \text{Obj } \underline{\mathcal{C}}$ .*

*I. The following conditions are equivalent.*

1.  *$M$  is weight-degenerate (resp.  $t(M) \smile^w 0$ ).*
2.  *$M$  can be presented as an extension of a left  $w$ -degenerate object of  $\underline{\mathcal{C}}$  by a right degenerate one (resp. by an element of  $\underline{\mathcal{C}}'_{w' \leq 0}$ ) in some weight-Karoubian extension  $\underline{\mathcal{C}}'$  of  $\underline{\mathcal{C}}$ .*
3. *Such a decomposition of  $M$  exists in any weight-Karoubian extension of  $\underline{\mathcal{C}}$ .*
4.  *$M$  is a  $\underline{\mathcal{C}}$ -retract of an extension of a left degenerate object of  $\underline{\mathcal{C}}$  by a right degenerate one (resp. by an element of  $\underline{\mathcal{C}}_{w \leq 0}$ ).*
5. *The object  $M \oplus M[-1]$  is an extension of this sort.*

*II. The following conditions are equivalent also.*

1.  *$t(M) \smile_w 0$ .*
2.  *$t(M)$  is a retract of a complex concentrated in non-positive degrees (in  $K(\underline{H}w)$ ).*
3.  *$M$  can be presented as an extension of an element of  $\underline{\mathcal{C}}'_{w' \geq 0}$  by a right degenerate object in some weight-Karoubian extension  $\underline{\mathcal{C}}'$  of  $\underline{\mathcal{C}}$ .*
4. *Such a decomposition of  $M$  exists in any weight-Karoubian extension of  $\underline{\mathcal{C}}$ .*
5.  *$M$  is a  $\underline{\mathcal{C}}$ -retract of an extension of an element of  $\underline{\mathcal{C}}_{w \geq 0}$  by a right degenerate object.*
6. *The object  $M \oplus M[1]$  is an extension of this sort.*

*Proof.* We will only prove assertion II; the proof of assertion I is similar.

Certainly, condition 6 of the assertion implies condition 5. 3 follows from 4 since a weight-Karoubian extension  $(\underline{\mathcal{C}}', w')$  of  $\underline{\mathcal{C}}$  exists (see Proposition 3.1.2(2)).

Next we note that (for any weight-Karoubian extension  $\underline{C}'$  of  $\underline{C}$  and a fixed  $M$ )  $t(M) \sim_w 0$  in  $K(\underline{Hw})$  if and only if this is true in  $K(\underline{Hw}')$  (see Proposition 1.3.2(6) and Remark 1.3.3(3,5)). Hence condition 5 implies condition 1. Besides, 1 is equivalent to 2 by Remark 1.3.3(5).

Next we fix some  $(\underline{C}', w')$  and recall that (the conclusion of) Theorem 2.3.4(II) can be applied to  $\underline{C}'$  according to Proposition 3.1.3(1). Hence condition 1 implies condition 4.

It remains to deduce condition 6 from condition 3. For any  $N' \in \text{Obj } \underline{C}'$  being the "formal image" of an idempotent  $p \in \underline{C}(N, N)$  (for some  $N \in \text{Obj } \underline{C}$ ) we have  $\text{Cone } p \cong N' \oplus N'[1] \in \text{Obj } \underline{C}$  (cf. Lemma 2.2 of [Tho97]). Hence the direct sum of the "decomposition" of  $N$  given by condition 3 with its shift by  $[1]$  yields condition 6.  $\square$

*Remark 3.1.6.* 1. Now we describe some consequences of our results that will be used in [Bon16].

Firstly, if  $\underline{C}$  is left non-degenerate then part I of our corollary certainly implies that any its  $w$ -degenerate object is right degenerate.

Now assume in addition that there is a weight-exact functor  $F : \underline{D} \rightarrow \underline{C}$ , where  $\underline{D}$  is a triangulated category endowed with a weight structure  $v$ . Consider two choices  $(M_1^i)$  and  $(M_2^i)$  of  $v$ -weight complexes of an object  $M$  of  $\underline{D}$ . If  $F$  kills all  $\underline{D}$ -morphisms from  $M_1^i$  and  $M_2^i$  for all  $i \in \mathbb{Z}$  then we certainly have  $t_w(F(id_M)) = 0$ . Hence  $F(M)$  is a right degenerate object of  $\underline{C}$ .

We will apply this statement for "computing intersections" of triangulated  $\underline{D}_1, \underline{D}_2 \subset \underline{D}$ . We assume that  $v$  restricts to  $\underline{D}_1$  and  $\underline{D}_2$  (so, this yields the corresponding choices of  $t_v(M)$  for  $M \in \text{Obj } \underline{D}_1 \cap \text{Obj } \underline{D}_2$ ). Thus if we assume in addition that a weight-exact  $\underline{D} \rightarrow \underline{C}$  annihilates all  $\underline{D}$ -morphisms from  $\underline{H}_1$  into  $\underline{H}_2$  (where  $\underline{H}_i$  are the hearts of the weight structures for  $\underline{D}_i$ ) then we will obtain that  $F(M)$  is right degenerate in  $\underline{C}$  for any  $M \in \text{Obj } \underline{D}_1 \cap \text{Obj } \underline{D}_2$ . In particular, if  $M$  is also  $v$ -bounded below then  $F(M) = 0$ . We will use this statement for  $F$  being the Verdier localization functor of  $\underline{D}$  by its Karoubi-closed subcategory  $\underline{D}_3$ ; so we obtain that  $M$  essentially (i.e., up to an isomorphism) belongs to  $\text{Obj } \underline{D}_3$  (if  $M$  is  $v$ -bounded below).

2. By Proposition 8.1.1 of [Bon10a] (cf. also Proposition 4.1.2 of [Sos15]) if  $\underline{D}$  is a small full (triangulated) subcategory of  $\underline{C}$  all of whose objects are  $w$ -degenerate and  $w$  restricts to it (i.e., any object of  $\underline{D}$  possesses a weight decomposition inside  $\underline{D}$ ) then the Verdier quotient  $\underline{E} = \underline{C}/\underline{D}$  possesses a

weight structure  $w_{\underline{E}}$  such that the localization functor  $l$  is weight-exact and  $\underline{Hw} \subset \underline{Hw}_{\underline{E}}$ . Hence  $l$  "does not affect" weight complexes (see Proposition 1.3.2(6)); thus the  $w$ -degenerate objects in  $\underline{C}$  are exactly the preimages of  $w_{\underline{E}}$ -degenerate objects in  $\underline{E}$ .

3. Now assume that  $\underline{D}$  is the whole subcategory of weight-degenerate objects (and it is small). Consider some weight-Karoubian extension  $\underline{C}'$  of  $\underline{C}$  (and denote by  $w'$  the corresponding weight structure). By Corollary 3.1.5(I),  $\underline{D}' = \text{Kar}_{\underline{C}'} \underline{D}$  (see §1.1) is exactly the subcategory of  $w'$ -degenerate objects in  $\underline{C}'$ . Next (as we have just shown)  $\underline{E}' = \underline{C}'/\underline{D}'$  is endowed with a weight structure such that the embedding  $\underline{C} \rightarrow \underline{E}'$  is weight-exact and  $\underline{Hw}_{\underline{E}'}$  is equivalent to  $\text{Kar}(\underline{Hw})$ . This weight-structure "restricts" to  $\underline{E}$  since all objects of the latter category possess weight decompositions that "come from"  $\underline{C}$  (cf. Proposition 8.1.1 of [Bon10a] again). Thus (see Theorem 2.3.4(III))  $\underline{E}$  possesses a non-degenerate weight structure such that  $l$  is weight-exact.

3. The author does not know whether one can always localize by degenerate objects if  $\underline{C}$  is not small (i.e., whether the hom-sets in the localized category are necessarily small; note still that for  $SH$  the localization by degenerate objects exists since it can be described as a Bousfield localization). Yet note that there exists a well-defined equivalence relation on  $\text{Obj } \underline{C}$  corresponding to the isomorphism of objects in this localization; one can certainly check whether an object is without weights  $m, \dots, n$  using this equivalence relation.

## 3.2 On pure functors that detect weights

Now we prove that "weights may be detected" using (co)homology of a certain sort.

Since in this section we will mostly study  $\underline{Hw}$ -complexes, our arguments will be rather easy exercises in basic homological algebra (along with a little of category theory). Yet their conclusions become quite interesting when combined with the following statement.

**Proposition 3.2.1.** *Let  $G : \underline{Hw} \rightarrow \underline{A}$  be an additive functor, where  $\underline{A}$  is any abelian category. Choose a weight complex  $t(M) = (M^j)$  for each  $M \in \text{Obj } \underline{C}$ , and denote by  $H(M) = H^G(M)$  the zeroth homology of the complex  $G(M^j)$ . Then  $H(-)$  yields a homological functor that does not depend on the choices of weight complexes. Moreover, the assignment  $G \mapsto H^G$  is natural in  $G$ .*

*Proof.* This is (most of) Corollary 2.3.4 of [Bon13] (combined with categorical duality).  $\square$

*Remark 3.2.2.* 1. We will call a functor  $H : \underline{C} \rightarrow \underline{A}$  *pure* (or *w-pure*) if it equals  $H^G$  for a certain  $G$ .

For any  $i$  we will denote the composite functor  $H \circ [i]$  by  $H_i$ .

2. Certainly (for any  $m \in \mathbb{Z}$ ) if an object  $M$  of  $\underline{C}$  is a retract of an extension  $M'$  of an element of  $\underline{C}_{w \geq -m}$  (resp. of  $\underline{C}_{w \leq -m}$ ) by a weight-degenerate object, then  $H_i^G(M) = H_i^G(M') = 0$  for any  $G$  and all  $i > m$  (resp. for all  $i < m$ ). The main goal of this subsection is to establish certain converse statements.

3. Loc. cit. also contains the following simple intrinsic description of pure functors (in the dual form):  $H$  is *w-pure* whenever its restrictions both to  $\underline{C}_{w \geq 1}$  and to  $\underline{C}_{w \leq -1}$  are zero.

We will not use this statement below. Note however that it certainly implies the following: a cohomological functor  $H : \underline{C} \rightarrow \underline{A}$  is of weight range  $[m, m]$  (for  $m \in \mathbb{Z}$ ; see Definition 1.4.1(3) above) whenever it has the form  $M \mapsto H_{-m}(G(M^*))$  (for some additive  $G : \underline{Hw}^{op} \rightarrow \underline{A}$ ).

Now we prove that certain pure functors detect weights of objects of  $\underline{C}$ . We will start from formulating a general statement of this sort; next we will study more concrete cases.

**Proposition 3.2.3.** *For  $G$  and  $\underline{A}$  as above assume that the following conditions are fulfilled:*

- (i) *the image of  $G$  consists of  $\underline{A}$ -projective objects only*
- (ii) *if an  $\underline{Hw}$ -morphism  $h$  does not split (i.e., it is not a retraction) then  $G(h)$  does not split also.*

*Then for any  $M \in \text{Obj } \underline{C}$ ,  $m \in \mathbb{Z}$ , and  $H = H^G$  we have the following:  $M$  is *w-bounded below* and  $H_i(M) = 0$  for all  $i > m$  if and only if  $M \in \underline{C}_{w \geq -m}$ .*

*Proof.* As we have noted in Remark 3.2.2(2), it suffices to prove the "only if" parts of our assertion.

Next, according to Propositions 3.1.2 and 3.1.3 we may assume that  $\underline{Hw}$  is Karoubian.

Moreover, we can assume that  $M \in (\underline{C}_{w \geq -n} \setminus \underline{C}_{w \geq -n+1})$  for some  $n \in \mathbb{Z}$ . Thus we can assume that its weight complex  $(M^i)$  is concentrated in degrees  $\leq n$  (see Proposition 1.3.2(3)). Moreover, the boundary morphism  $d_M^{n-1} : M^{n-1} \rightarrow M^n$  cannot split (easy from Corollary 3.1.5(II)). Thus  $G(d_M^{n-1})$  does

not split also. Since  $G(M^n)$  is projective, this non-splitting implies that  $H_n(M) \neq 0$ . Thus  $n \leq m$ .  $\square$

*Remark 3.2.4.* 1. Certainly, condition (ii) of the proposition is fulfilled both for  $G$  and for the opposite functor  $G^{op} : \underline{Hw}^{op} \rightarrow \underline{A}^{op}$  whenever  $G$  is a full embedding. So, it may be interesting to assume (in addition) that the image of  $G$  consists of injective objects only.

2. Now we describe a general method for constructing a full embedding  $G$  whose image consists of  $\underline{A}$ -projective objects.

Assume that  $\underline{Hw}$  is an essentially small (additive)  $R$ -linear category, where  $R$  is a commutative unital ring (certainly, one may take  $R = \mathbb{Z}$  here). Denote some small skeleton of  $\underline{Hw}$  by  $B$ .

Consider the abelian category  $\text{PShv}^R(B)$  of  $R$ -linear contravariant functors from  $B$  into the category of  $R$ -modules. Then  $B$  (and so, also  $\underline{Hw}$ ) embeds into the full subcategory of projective objects of  $\text{PShv}^R(B)$  (by the Yoneda lemma; see Lemma 8.1 of [MVW06]).

3. The objects in the image of this functor may be called *purely  $R$ -representable homology*. Since they are usually not injective in  $\text{PShv}^R(B)$ , a dual construction may be useful for checking whether  $M \in \underline{C}_{w \leq -m}$ .

4. Condition (ii) is certainly necessary for our proposition. Indeed, if  $h \in \text{Mor}(\underline{Hw})$  does not split whereas  $G(h)$  does, then one can easily check that  $\text{Cone}(h) \in \underline{C}_{w \geq 0} \setminus \underline{C}_{w \geq 1}$  and  $H_i(\text{Cone}(h)) = 0$  for  $i \neq -1$ .

5. One may say that the functors we consider "detect weights" (from below or from above, respectively). This notion is closely related to the one of *weight-conservativity* introduced in [Bach15].

**Proposition 3.2.5.** *Assume that  $G : \underline{C} \rightarrow \underline{A}$  is a full embedding and  $\underline{A}$  is semi-simple. Then  $H_i(M) = 0$  for all  $i > m$  (resp. for all  $i < m$ ) if and only if  $M \oplus M[1]$  is an extension of an element of  $\underline{C}_{w \geq -m}$  by a right degenerate object (resp.  $M \oplus M[-1]$  is an extension of an element of  $\underline{C}_{w \leq -m}$  by a left degenerate object).*

*Proof.* Once again, we may assume that  $\underline{Hw}$  is Karoubian (see Corollary 3.1.5); thus it is abelian semi-simple itself. Then the vanishing of  $H_i(M) = 0$  for all  $i > m$  (resp. for all  $i < m$ ) certainly yields that  $t(M)$  is homotopy equivalent to a complex concentrated in degrees at most  $m$  (resp. at least  $m$ ). Thus it remains to apply Corollary 3.1.5(II) once more.  $\square$

*Remark 3.2.6.* Recall that the orthogonality axiom (in Definition 1.2.1(I)) immediately yields that  $w$  is non-degenerate whenever it is bounded.

Now, if  $w$  is non-degenerate then under the assumptions of Proposition 3.2.5 we obtain that  $M \in \underline{C}_{w \leq -m}$  whenever  $H_i(M) = 0$  for all  $i < m$ . Hence in this setting the vanishing of  $H_i(M) = 0$  for all  $i \neq m$  is equivalent to  $M \in \underline{C}_{w=-m}$ .

### 3.3 Some counterexamples in the non-Karoubian case

Our examples will be rather simple; their main "ingredient" is  $K(L - \text{vect})$  (the homotopy category of complexes of finite dimensional  $L$ -vector spaces; here  $L$  is an arbitrary fixed field).

#### 3.3.1 An "indecomposable" weight-degenerate object

Now we demonstrate that Theorem 2.3.4(II) does not extend to arbitrary (i.e., to not necessarily weight-Karoubian) triangulated categories.

Our example will be the full subcategory  $\underline{C}$  of  $(K^b(L - \text{vect}))^3$  consisting of objects whose "total Euler characteristic" is even (i.e., the sum of dimensions of all cohomology of all the three components of  $M = (M_1, M_2, M_3)$  should be even). We define  $w$  for  $\underline{C}$  as follows:  $\underline{C}_{w \leq 0}$  consists of those  $(M_1, M_2, M_3)$  such that  $M_1 \cong 0$  and  $M_2$  is acyclic in negative degrees;  $(M_1, M_2, M_3) \in \underline{C}_{w \geq 0}$  if  $M_1 \cong 0$  and  $M_2$  is acyclic in positive degrees. This is easily seen to be a weight structure (in particular, a weight decomposition of  $(M_1, M_2, M_3)$  is  $(0, M', M_3) \rightarrow (M_1, M_2, M_3) \rightarrow (M_1, M'', 0)$ , where  $M' \rightarrow M_2 \rightarrow M''$  is a stupid weight decomposition of  $M_2$  in  $K^b(L - \text{vect})$  with the corresponding parity of the Euler characteristics). Next, one can easily see that the object  $M = (L, 0, L)$  (here we put the  $L$ 's in degree 0 though the degrees make no difference) is weight-degenerate (since it is weight-degenerate in the obvious extension of  $w$  onto its weight-Karoubian extension  $\underline{C}' = (K^b(L - \text{vect}))^3$ ; see Proposition 1.3.2(6)). Yet  $M$  certainly cannot be presented as an extension of a left degenerate object (i.e., of an object whose last two components are zero) by an element of  $\underline{C}_{w \leq 0}$  (since the corresponding "decomposition" in  $\underline{C}'$  is unique and its "components" have odd "total Euler characteristics"). So, we obtain that first two statements in Theorem 2.3.4(II) do not extend to  $\underline{C}$ ; for the same reasons, the third statement in loc. cit. does not extend to  $\underline{C}$  either (and the same  $M$  does not possess the corresponding "decomposition").



Looking at the proof Theorem 2.3.4(II) one immediately obtains the existence of  $n > 0$  such that there does not exist a triangle  $X_n \rightarrow M \rightarrow Y_n$  with  $X_n \in \underline{C}_{w \leq -n}$ ,  $Y_n \in \underline{C}_{w \geq n}$ . Moreover, one can easily check directly that a triangle of this sort does not exist for  $n = 1$  already.

### 3.3.2 A bounded object that is without weight 0 but does not possess a decomposition avoiding this weight

So, the example above yields that Theorem 2.2.1(8) does not extend to arbitrary  $(\underline{C}, w)$  (i.e., that our definition of objects without weights  $m, \dots, n$  is not equivalent to Definition 1.10 of [Wil09] in general). Yet the weight structure is degenerate in this example. Now we give a bounded example of the non-equivalence of definitions. Denote by  $\underline{B}$  the category of even-dimensional

vector spaces over  $L$ ; take  $\underline{C} = K^b(\underline{B})$ ,  $M = L^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} L^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} L^2$ ; we put these vector spaces in degrees  $-1, 0$ , and  $1$ , respectively. Certainly, the composition  $(L^2 \rightarrow L^2 \rightarrow 0) \rightarrow M \rightarrow (0 \rightarrow L^2 \rightarrow L^2)$  is zero; so,  $M$  is without weight 0 (see Proposition 2.1.1(1)). Yet  $M$  does not possess a decomposition avoiding weight 0 since the  $L$ -Euler characteristics of the corresponding  $X$  and  $Y$  cannot be odd.

Certainly, this example also yields that decompositions avoiding weights  $m, \dots, n$  do not "lift" from a (weight-)Karoubian  $\underline{C}'$  (in our case  $\underline{C}' = K^b(L\text{-vect})$ ); the corresponding weight structure is the stupid one) to  $\underline{C}$  (cf. Remark 2.2.2(5)).

## References

- [Ayo15] Ayoub J., Motives and algebraic cycles: a selection of conjectures and open questions, preprint, 2015, <http://user.math.uzh.ch/ayoub/PDF-Files/Article-for-Sтивен.pdf>
- [Bach15] T. Bachmann, On the invertibility of motives of affine quadrics, preprint, <http://arxiv.org/abs/1506.07377>
- [BaS01] Balmer P., Schlichting M. Idempotent completion of triangulated categories// Journal of Algebra 236(2), 2001, 819–834.

- [Bar05] Barr M., Absolute homology// Theory and Applications of Categories vol. 14, pp. 53–59, 2005.
- [BBD82] Beilinson A., Bernstein J., Deligne P., Faisceaux pervers, *Asterisque* 100, 1982, 5–171.
- [BoK90] Bondal A. I., Kapranov M. M. Enhanced triangulated categories. (Russian)// *Mat. Sb.* 181 (1990), no. 5, 669–683; translation in *Math. USSR-Sb.* 70 (1991), no. 1, 93–107.
- [Bon09] Bondarko M.V., Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky vs. Hanamura// *J. of the Inst. of Math. of Jussieu*, v.8 (2009), no. 1, 39–97, see also <http://arxiv.org/abs/math.AG/0601713>
- [Bon10a] Bondarko M., Weight structures vs.  $t$ -structures; weight filtrations, spectral sequences, and complexes (for motives and in general)// *J. of K-theory*, v. 6, i. 03, pp. 387–504, 2010, see also <http://arxiv.org/abs/0704.4003>
- [Bon10b] Bondarko M.V., Motivically functorial coniveau spectral sequences; direct summands of cohomology of function fields// *Doc. Math.*, extra volume: Andrei Suslin’s Sixtieth Birthday (2010), 33–117; see also <http://arxiv.org/abs/0812.2672>
- [Bon13] Bondarko M.V., Gersten weight structures for motivic homotopy categories; direct summands of cohomology of function fields and coniveau spectral sequences, preprint, <http://arxiv.org/abs/1312.7493>
- [Bon14] Bondarko M.V., Weights for relative motives: relation with mixed complexes of sheaves// *Int. Math. Res. Notes*, vol. 2014, i. 17, 4715–4767; see also <http://arxiv.org/abs/1007.4543>
- [Bon15] Bondarko M.V., Mixed motivic sheaves (and weights for them) exist if ‘ordinary’ mixed motives do, *Compositio Mathematica*, vol. 151, 2015, 917–956.
- [Bon16] Bondarko M.V., Intersecting the dimension filtration with the slice one for (relative) motivic categories, preprint, <http://arxiv.org/abs/1603.09330>

- [BoI15] Bondarko M.V., Ivanov M.A., On Chow weight structures for *cdh*-motives with integral coefficients, *Algebra i Analiz*, v. 27 (2015), i. 14–40, see also <http://arxiv.org/abs/1506.00631>
- [BoS16] Bondarko M.V., Sosnilo V.A., On constructing weight structures and extending them onto idempotent extensions, preprint, <http://arxiv.org/abs/1605.08372>
- [BoT15] Bondarko M.V., Tabuada G., Picard groups, weight structures, and (noncommutative) mixed motives, preprint, <http://arxiv.org/abs/1512.09101>
- [Chr98] Christensen J., Ideals in triangulated categories: phantoms, ghosts and skeleta// *Advances in Mathematics* 136.2 (1998), 284–339.
- [Deg11] Déglise F., Modules homotopiques (Homotopy modules)// *Doc. Math.* 16 (2011), 411–455.
- [GiS96] Gillet H., Soulé C. Descent, motives and *K*-theory// *J. f. die reine und ang. Math.* v. 478, 1996, 127–176.
- [Heb11] Hébert D., Structures de poids a la Bondarko sur les motifs de Beilinson// *Compositio Mathematica*, vol. 147, is. 5, 2011, 1447–1462.
- [Mar83] Margolis H.R., Spectra and the Steenrod Algebra: Modules over the Steenrod Algebra and the Stable Homotopy Category, Elsevier, North-Holland, Amsterdam-New York, 1983.
- [MVW06] Mazza C., Voevodsky V., Weibel Ch., Lecture notes on motivic cohomology, Clay Mathematics Monographs, vol. 2, 2006.
- [Pau08] Pauksztello D., Compact cochain objects in triangulated categories and co-t-structures// *Central European Journal of Mathematics*, vol. 6, n. 1, 2008, 25–42.
- [PoS16] Pospisil D, Stovicek J., On compactly generated torsion pairs and the classification of co-t-structures for commutative noetherian rings// *Trans. Amer. Math. Soc.*, 368 (2016), 6325–6361.
- [Sch11] Schnürer O., Homotopy categories and idempotent completeness, weight structures and weight complex functors, preprint, <http://arxiv.org/abs/1107.1227>

- [Sos15] Sosnilo V.A., Weight structures in localizations (revisited) and the weight lifting property, preprint, <http://arxiv.org/abs/1510.03403>
- [Tho97] Thomason R.W., The classification of triangulated subcategories// *Comp. Math.*, vol. 105, iss. 01, 1997, 1–27.
- [Wil09] Wildeshaus J., Chow motives without projectivity// *Compositio Mathematica*, v. 145(5), 2009, 1196–1226.
- [Wil08] Wildeshaus J., Notes on Artin-Tate motives, preprint, <http://www.math.uiuc.edu/K-theory/0918/>, to appear in: Bost, J.-B., Fontaine, J.-M. (eds.) *Autour des motifs—Ecole d’été Franco-Asiatique de Géométrie Algébrique et de Théorie des Nombres. Vol. III, Panoramas et Synthèses*, Soc. Math. France.
- [Wil15a] Wildeshaus J., On the interior motive of certain Shimura varieties: the case of Picard surfaces// *Manuscripta Math.*, vol. 148, i. 3, 2015, 351–377.
- [Wil15b] Wildeshaus J., Weights and conservativity, preprint, <http://arxiv.org/abs/1509.03532>